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Stability of viscous lubricated thin film down an inclined plane beneath ambient lighter non miscible static liquid

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ABSTRACT

This paper considers the stability of a thin film propagating beneath a large quantity of ambient static non miscible lighter liquid and over a sloping plane. Such configuration that has never been considered earlier can model the spill of a heavy hydrocarbon into the ocean by a tanker. Equations of conservation of the mass and the momentum were appropriately made non dimensional and a similar solution is proposed in this paper. In this way, an analytical expression of the hydrodynamic field, say velocity field and pressure field is provided. Then, the equation governing the spatiotemporal evolution of the water-oil interface was built and solved by a perturbation method. Also, the time evolution of the wave front position along the inclined plane was built. Finally, the effect of the control parameters on the linear stability of the flow was investigated.

Keywords: *Gravity Current, Hydrodynamic stability, Liqui-liquid interface*

1. Introduction

Oil spills can be caused by the accidental (ruin of the material constituting the tank) or intentional (degassing or act of war) spillage of a large quantity of oil from an oil tanker into the ocean. The collection, destruction, and storage of the oil that ran aground on the coast, as well as the coast cleaning are arduous, dangerous and expensive tasks for the populations living in these soiled territories (Tansel, 2014; French-McCay, 2004; Reed et al., 1999). The oil which is spilled in the ocean undergoes three main phenomena: entrainment on the ocean surface on the one hand and at the ocean bottom on the other hand and evaporation as well as physico-chemical reactions with the medium in which it evolves. Moreover, to date, no universal law makes it possible to determine a priori the respective proportions of oil in these three processes. Therefore, an oil spill is a complex phenomenon of which each process must firstly be controlled before they are put into competition by a relevant modeling. In this paper, we study the 1D propagation of a fixed volume of dense fluid (with density ρ_2 and kinematic viscosity ν_2) released from a reservoir to the bottom of a large body of ambient lighter non miscible liquid, static water in this case (with density ρ_1 and kinematic viscosity ν_1). Based on this description, the phenomenon studied is defined as a lubricated (a flow in which one dimension, thickness in this case, is significantly smaller than the others) viscous (the inertial terms in the momentum equation can be neglected versus the viscous ones) film flow.

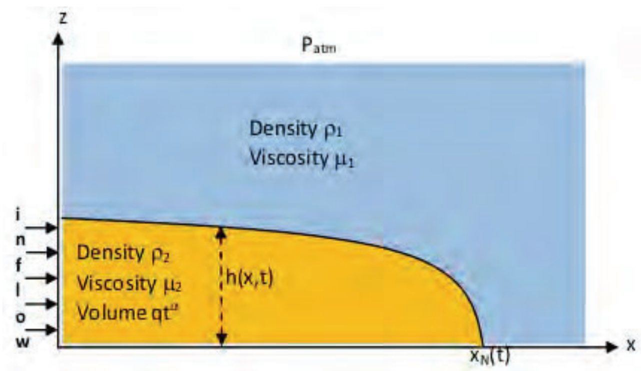

Figure 1. Intrusion of a dense liquid in a lighter ambient liquid upon a horizontal plane (Huppert, 1982)

The ocean is not flat so, for modelling purpose in the present pioneering study, a basic configuration where the ocean bottom is an inclined plane is considered as a basic case and is not applied to a particular real ocean bottom topography. Consequently, the flow is controlled by gravity, oil/water density difference, oil viscosity and surface tension.

The lubricated gravity flow of one liquid in another liquid upon a horizontal plane (termed as gravity current) was investigated earlier in a unidirectional configuration (figure 1) by Fay (1969), Hoult (1986), Britter (1979), Didden & Maxworthy (1982) and Huppert (1982), Kowal & Worster (2019).

In lubrication theory, the Navier Stokes equations describing the one-dimensional lubricated intrusion (in x-direction) of a dense gravity current beneath a static large volume of lighter static ambient liquid reduce to its projection in the x-direction in the following form :

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} \tag{1}$$

where the longitudinal pressure gradient in the current can be written

$$\frac{\partial p}{\partial x} = (\Delta\rho) g \frac{\partial h(x,t)}{\partial x} \tag{2}$$

with $\Delta\rho = \rho - \rho_0$ (3)

Inserting Equ.(2) in Equ.(1), we get

$$0 = -\frac{\Delta\rho}{\rho} g \frac{\partial h}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} \tag{4}$$

Self-similar solutions are then sought in the following form where x_N is the abscissa of the current front.

$$x_N = Kt^{\beta} \; ; \; h = \frac{q}{K} t^{\alpha-\beta} f\left(\frac{x}{x_N(t)}\right) \tag{5}$$

Hydrodynamic stability is a fundamental topic of the fluid dynamics curriculum in schools of engineering. A laminar flow is said to be stable if a small disturbance superimposed on it vanishes over time. The flow is unstable if this small disturbance increases or remains constant over time. The aim of this work is to point out the control parameters that promote instability in order to provide reliable information to contribute to the protection of the ocean. Since Kapitza's original work on stability of film flow over an inclined plane (Kapitza &Kapitza, 1949), many papers have been published on this topic (e.g.: Yih, 1955; 1963, Benjamin, 1957, Kao, 1964 among others). They found that the critical Reynolds number Re_c, i.e. the threshold above which some disturbances will be amplified depends only on the slope and is given by

$$\mathrm{Re}_c = \frac{5}{6}\cot\theta \tag{6}$$

Moreover, very short waves are damped by surface tension. Kao (1964) extended that basic configuration to flow of a binary system of two layers of viscous fluids of different densities. More recently, the shallow water models (e.g. : Ruyer-Quil & Manneville, 1998 ; 2000) provide a good understanding of the stability of Newtonian fluids. For power-law fluids, Ng. & Mei (1994) as well as Hwang & al. (1994) built lubrication models while Nsom et al. (2019) proposed a generalized Orr-Sommerfeld model with appropriate definition of non-dimensional numbers. All of the models existing in literature were restricted to the case where the flow develops in the atmosphere, while in the present paper, the case where the ambient fluid is a large volume of static non miscible liquid is tackled.

The paper is organized as follows. In the second section, the flow configuration is presented and the equations of motion are presented and made non-dimensional. These equations of motion are solved and the hydrodynamic field as well as the interface profile are derived in the third section. The fourth section is devoted to the linear stability where Orr-Sommerfeld equation is built and solved by the method of perturbation and notably the effects of the different control parameters on the linear stability of the flow are respectively pointed out. Finally, the fifth presents a discussion and the conclusions.

2. Problem statement

2.1. Flow configuration

We consider a fixed quantity of a dense viscous Newtonian fluid (heavy crude oil) spilled at time $t = 0^-$ upon an inclined plane in the form of a thin film beneath a large quantity of static lighter non miscible liquid (sea water in this case) between abscissa film tail $(x = -l)$ and film front $(x = 0)$. Downstream, the film front is occupied by the ambient static liquid. At initial time, $t = 0^+$ the film flows downstream the sloping plane with slope θ and length L, with $L >> l$ and beneath the ambient static liquid. The width of the flow system is infinite.

The quantities referring to lighter ambient static liquid have the subscript 1, while those referring to the heavy viscous liquid contained in the reservoir at negative time and that will undergo the flow have the subscript 2. Typically, we have $\rho_2 > \rho_1$. The flow configuration is presented on figure 2.

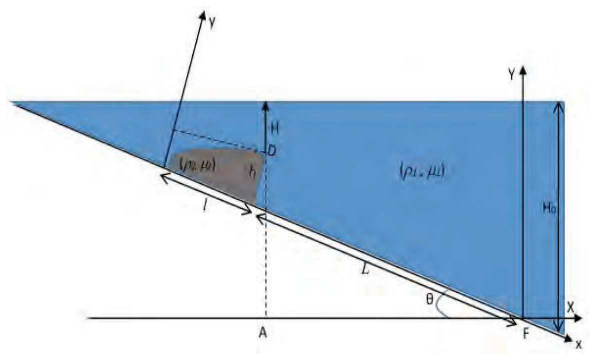

Figure 2. Flow configuration

2.2. Equations of motion

If $V_b^*(U_b^*,0,0)$ and P_b^* denote the velocity field and the pressure field in the gravity current that propagates downstream the inclined plane, the 1D flow in hand is governed by the conservation equations of the mass (Equ.(7)) and of the momentum (Equs.(8)-(9)). They write:

$$\frac{\partial U_b^*}{\partial x^*} = 0 \quad (7)$$

which brings that $U_b^* = U_b^*(y)$

$$\rho_2 \left(U_b^* \frac{\partial U_b^*}{\partial x^*} \right) = -\frac{\partial P_b^*}{\partial x^*} + \mu_2 \left(\frac{\partial^2 U_b^*}{\partial (x^*)^2} + \frac{\partial^2 U_b^*}{\partial (y^*)^2} \right) + \rho_2 g \sin(\theta) \quad (8)$$

$$0 = -\frac{\partial P_b^*}{\partial y^*} + \rho_2 g \cos(\theta) \quad (9)$$

3. Basic laminar flow

3.1. Hydrodynamic field

Equ.(9) shows that the pressure field is hydrostatic so it writes

$$P_b^*(y) = -\rho_2 g y^* \cos\theta + B \quad (10)$$

Denoting the atmospheric pressure by P_{atm}, the constant of integration B is determined by equating at the interface, the pressure in ambient liquid given by Pascal law as $P_1(\text{interface}) = \rho_1 gH + P_{atm}$ with its value in the denser liquid given by Equ.(10) as $P_2(\text{interface}) = B$. Consequently, the pressure field in the gravity current has the following expression:

$$P_b^*(y) = -\rho_2 g y^* \cos\theta + \rho_1 g\,[(H_0 - L\sin\theta) - h^*(x^*)\cos\theta] + P_{atm} \quad (11)$$

Introducing that expression of the pressure field in Equ.(8), we get

$$\mu_2 \frac{d^2 U_b^*}{d(y^*)^2} = -\rho_1 g \cos\theta \frac{\partial h^*}{\partial x^*} - \rho_2 g \sin\theta \quad (12)$$

whose general solution writes

$$\mu_2 U_b^*(y) = -g\frac{(y^*)^2}{2}\left[\rho_1 \cos\theta \frac{\partial h^*}{\partial x^*} - \rho_2 \sin\theta\right] + K_1 y + K_2 \quad (13)$$

The constants of integration K_1 and K_2 are determined by the boundary conditions. The first boundary condition is the no slip condition at channel bed, i.e.

$$U_b^* = 0 \quad \text{at} \quad y^* = -h^* \quad (14)$$

while the second boundary condition expresses the continuity of the shear stress at the interface. Assuming the ambient liquid (water) to be a perfect fluid, it writes

$$\mu_2 \frac{dU_b^*}{dy^*} = 0 \quad \text{at} \quad y^* = 0 \quad (15)$$

After straightforward calculations, the velocity field is obtained with the following expression

$$U_b^*(y) = -\frac{g}{2\mu_2}\left[\rho_1 \cos\theta \frac{dh^*}{dx^*} - \rho_2 \sin\theta\right][(y^*)^2 - (h^*)^2] \quad (16)$$

3.2. Evolution equation of the interface

To form the evolution equation of the interface, the continuity equation is written in its global form, i.e.

$$\frac{\partial h^*}{\partial t^*} + \frac{\partial}{\partial x^*}\left[\int_0^h U_b^*(y^2) dy^*\right] = 0 \quad (17)$$

Using the expression of the velocity field given by Equ.(16), the integral appearing in Equ.(17) writes

$$\int_0^h U_b^*(y) dy = -\frac{(\rho_2 - \rho_1) g \cos\theta}{3\mu_2}\left[(h^*)^3 - \frac{\partial h^*}{\partial x} - \frac{\sin\theta}{\cos\theta}(h^*)^3\right] \quad (18)$$

So, equation of continuity takes the form

$$\frac{\partial h^*(x^*,t^*)}{\partial t^*} - \frac{(\rho_2 - \rho_1) g \cos\theta}{12\mu_2}\left\{\frac{\partial^2}{\partial (x^*)^2}[(h^*)^4(x^*,t^*)] - 4tg\theta \frac{\partial}{\partial x^*}[(h^*)^3]\right\} = 0 \quad (19)$$

Assuming the following set of non-dimensional functions and variables in which the characteristic height of the gravity current H ; the characteristic abscissa is L and the characteristic longitudinal velocity U

$$x = \frac{x^*}{L}; U_b = \frac{U_b^*}{U}; U = \frac{(\rho_2-\rho_1)H^3 g\cos\theta}{12\mu_2 L}; h = \frac{h^*}{H} \quad (20)$$

with the lubrication assumption $H << L$ and the

characteristic time is derived as $T = \frac{L}{U}$

Equ.(18) takes the following non-dimensional form

$$\frac{\partial h}{\partial t} - \frac{\partial^2 [h^4]}{\partial x^2} + 4\lambda \frac{\partial [h^3]}{\partial x} = 0 \quad (21)$$

Parameter λ is defined as

$$\lambda = \frac{L}{H}\text{tg}\theta \quad (22)$$

3.3. Interface profile

3.3.1 Spatiotemporal equation of interface profile

Flow upon an inclined plane is governed by Eqs.(20) whose meaning is that the convective term is balanced by the summ of the two others. Meanwhile these two tems are not of the same order of magnitude. The order of magnitude of the second term of Equ.(21) is $\frac{H^4}{L^2}$ while the order of magnitude of the third term is $\frac{H^3}{L}$. From the lubrication assumption the second term of Equ.(21) is much smaller than the third one so, it can be neglected. Consequently, Equ.(21) becomes

$$\frac{\partial h}{\partial t}+4\lambda\frac{\partial\left[h^3\right]}{\partial x}=0 \qquad (23)$$

We seek similar solutions to Equ.(23) of the form

$$h(x,t)=D\cdot x^{\beta}\cdot t^{-\gamma} \qquad (24)$$

where D,β,γ are constants that we can determine by inserting the form assumed for the solution Equ.(24) in the equation of motion (Equ.(23) we find

$$\beta=\gamma=\tfrac{1}{2} \quad ; \quad D=\frac{1}{2\sqrt{3\lambda}} \qquad (25)$$

Inserting these values in Equ.(24), the interface evolution equation writes

$$h(x,t)=\frac{1}{2\sqrt{3\lambda}}\sqrt{\frac{x}{t}} \qquad (25)$$

Fig.3 and Fig.4 show for assigned parameter λ the variation of fluid height vs. abscissa at given time and vs. time at given abscissa, respectively. It can be noticed that the smaller parameter the greater fluid height. That flow characteristic is explained by the fact that the higher parameter λ, i.e. the higher the slope for given ratio H/L, the greater the gravity effect. Moreover, Fig. 3 exhibits the general space evolution of a gravity current. Immediately following the initiation of the flow, an inertial regime takes place where the inertia dominates the other effects present in the flow and equation of characteristic tangent is given by $h=4x$. During the development of the inertial regime where the equation of characteristic tangent is given by $h=0.75x$, the viscous effects increase to the point of becoming dominant. The viscous regime then settles over a given distance, until an equilibrium is reached which characterizes the asymptotic regime where the equation of characteristic tangent is given by $h=0.2x$.

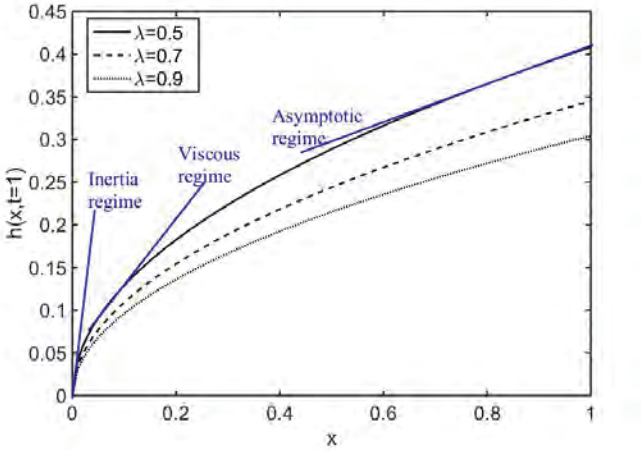

Figure 3. Variation of fluid height vs. abscissa at given time and for assigned parameter λ

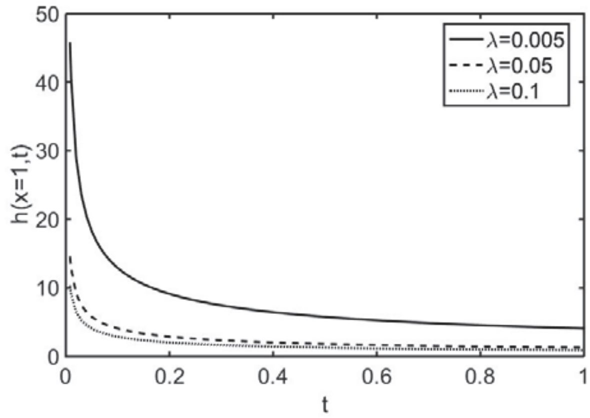

Figure 4. Variation of fluid height vs. time at given abscissa and for assigned parameter λ

3.3.2 *Law of evolution of the front*

To determine the law of evolution of the front abscissa x_f, we state that the fluid volume V is equal to a constant δ, that is equal (see Fig.2) at any time *t*, per unit width in non dimensional variable to

$$\delta=\frac{l}{2L} \qquad (26)$$

$$\frac{1}{2\sqrt{3\lambda t}}\int_0^{x_f}\sqrt{x}=\delta \qquad (27)$$

Taking one quadrature of Equ.(27), we get

$$x_f=Ct^{1/3} \qquad (28)$$

where

$$C=\left(3\lambda^{1/3}\delta^{2/3}\right) \qquad (29)$$

Fig. 5 shows, for assigned value to parameter λ, the time variation of the abscissa of film front vs. parameter δ. It can be noticed that the higher parameter δ, the greater front abscissa. That flow characteristic is explained by the fact that the higher parameter δ, the greater the liquid volume and consequently the faster the flow.

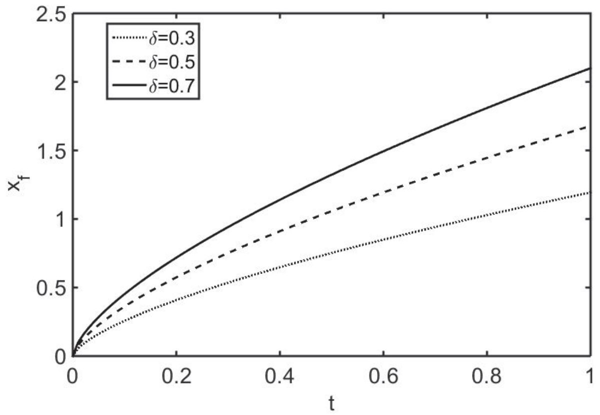

Figure 5. Time evolution of front abscissa for given parameters λ and δ

3.3.3 *Interface profile*

Combining the results of the two previous sections, the whole interface can be shown. A master curve is obtained for given parameter λ and the front is obtained to close the interface downwards after computing front abscissa, for assigned parameter δ. As shown in Fig.6 where the interfaces corresponding to three flow configurations from the same master curve obtained with $\lambda = 0.5$ and the respective fronts can be drawn, considering the corresponding value of parameter δ, namely 0.3 ; 0.5 and 0.7 in this case

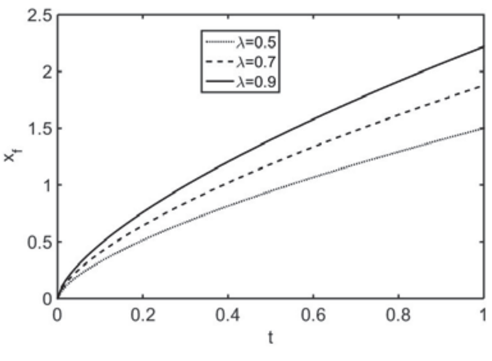

Figure 6. Interface profile

4. Linear stability analysis

4.1. The Orr-Sommerfeld equation

If, from Equ.(16), we take the surface velocity as the reference velocity U_0, i.e.

$$U_b^*(y=0) = \frac{g}{2\mu_2}\left[\rho_1 \cos\theta \frac{dh^*}{dx} + \rho_2 \sin\theta\right]\left[(h^*)^2\right] \tag{30}$$

Then, in non dimensional form, the velocity field of the basic flow field writes

$$U(y) = 1 - y^2 \tag{31}$$

where the flow depth has been used as reference length defining the non dimensional normal coordinate y.

To investigate the linear stability of the previous laminar steady basic flow, a small 2D perturbation with velocity field $[u(x,y,t)]$; $[v(x,y,t)]$ and pressure field $[p(x,y,t)]$ is superimposed to the above steady basic flow. Using the characteristic length, velocity and time defined in Equ.(20), the equations of conservation of mass and momentum are written in non dimensional form for the resulting hydrodynamic field with

$$\left[\vec{U}_r\left(U_p, V_p\right), P_p\right]$$

$$U_p = U_b(y) + u(x,y,t);\ V_p = v(x,y,t);$$

$$P_p = P_b(x,y) + p(x,y,t) \tag{32}$$

in the following form:

$$\frac{\partial U_p}{\partial x} + \frac{\partial V_p}{\partial y} = 0 \tag{33}$$

$$\frac{\partial U_p}{\partial t} + U_p \frac{\partial U_p}{\partial x} + V_p \frac{\partial U_p}{\partial y} = -\frac{\partial P_p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 U_p}{\partial x^2} + \frac{\partial^2 U_p}{\partial y^2}\right) + \frac{1}{Fr^2}\sin\theta \tag{34}$$

$$\frac{\partial V_p}{\partial t} + U_p \frac{\partial V_p}{\partial x} + V_p \frac{\partial V_p}{\partial y} = -\frac{\partial P_p}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 V_p}{\partial x^2} + \frac{\partial^2 V_p}{\partial y^2}\right) + \frac{\cos\theta}{Fr^2} \tag{35}$$

where Reynolds number, Froude number respectively defined as

$$\mathrm{Re} = \frac{\rho_2 L U}{\mu_2} \quad ; \quad Fr = \frac{U}{\sqrt{gL}} \tag{36}$$

have been sorted as non dimensional control parameters of the flow.

The disturbance superimposed to the basic laminar steady flow produces a deformation $\hat{\eta}(x,y,t)$ of the interface. To satisfy identically the continuity equation, we introduce a stream function $\hat{\psi}(x,y,t)$ according to :

$$u(x,y,t) = \frac{\partial \hat{\psi}(x,y,t)}{\partial y} \quad ; \quad v(x,y,t) = \frac{\partial \hat{\psi}(x,y,t)}{\partial x} \tag{37}$$

Assuming that the instability sets with respect to long waves as in the case where the flow develops in open atmosphere (Liu et al., 1993 ; Smith, 1990 ; Kelly & Goussis, 1989), we investigate the linear regime where the waves are sinusoidal, where the stream function, the perturbation on the free surface and the pressure field can be expanded in normal modes repectively with the form :

$$\hat{\psi} = \psi(y)e^{i\alpha(x-ct)} \quad ; \quad \hat{\eta} = \eta(y)e^{i\alpha(x-ct)} \quad ;$$

$$\hat{p} = p(y)e^{i\alpha(x-ct)} \tag{38}$$

We consider a temporal analysis where the wave number α is real and is viewed as a small parameter, while the wave speed c is complex with the form $c = c_r + ic_i$ and i is the imaginary unit defined by $i^2 = -1$.

Substituting the disturbance hydrodynamic field described by Equs.(37)-(38) in the equations of motion (Equs.(33)-(34)), we get after a straightforward handling:

$$\psi'''' - 2\alpha^2\psi'' + \alpha^4\psi - i\alpha \mathrm{Re}\left[(U-c)\left(\psi'' - \alpha^2\psi\right) - U''\psi\right] \tag{39}$$

Equ.(39) known as the Orr-Sommerfeld equation (Orr, 1907; Sommerfeld, 1908) governs the stability of a Newtonian fluid over a sloping plane. The effect of the ambient liquid, water in this case, is described by the following boundary conditions.

a/ The no slip condition

$$u = 0 \quad \text{and} \quad v = 0 \quad \text{at} \quad y = 0 \tag{40}$$

Using Equs.(37), we get in terms of stream function:

$$\psi' = 0 \quad \text{and} \quad \psi = 0 \quad \text{at} \quad y = 0 \tag{41}$$

b/ The kinematic condition

$$\psi(0) - (c-1)\eta = 0 \tag{42}$$

c/ The dynamic condition

$$\psi''(0) + \alpha^2\psi(0) + \eta D^2(0) = 0 \tag{43}$$

$$\psi'''(0) + \left(3\alpha^2 - i\alpha Re(c-1)\right)\psi^0(0) + i\alpha Re\left(\frac{1}{F_r} + \frac{\alpha^2}{We}\right)\eta = 0 \tag{44}$$

where a fifth characteristic non dimensional quantity We^* and which can be turned as a characteristic Weber number, defined as

$$We^* = \frac{\rho_0 U_0 L_0}{\gamma_0} \tag{45}$$

Has been introduced, while the following relation exists between the Reynolds number and the Froude number

$$Fr^* = \frac{Re^* tan\beta}{2} \tag{46}$$

In the next sections, Orr-Sommerfeld equation given by Equ.(39) with the associated boundary conditions governed by Equs.(41); (42) ; (43) ; (44) can be solved analytically, using a perturbation method. In that analysis, the velocity field of the basic flow in non dimensional form was obtained in Equ.(38) and it is assumed following Charru (2007)that $\alpha^2/We = O(1)$. Furthermore, the celerity and the stream function can be investigated in the form of power series of that small parameter, say: $c = c_0 + \alpha c_1$ and $\psi(y) = \psi_0(y) + \alpha\psi_1(y)$ respectively. The index gives the order of the solution.

In fact, Nsom et al. (2019) proposed a generalized model for shear thinning fluid over an inclined plane surrounded by the atmosphere that reduces to the previous model if power-law index equals unity $(n = 1)$, i.e. for Newtonian fluid. So, a similar solution can be derived, provided that in the solution built by Nsom et al. (2019), we put $(n = 1)$. Notably, the following results can be derived, where the subscript on the non dimensional parameters Re, Fr, We will be dropped, for making the notations simpler.

5. Solution

5.1. General solution

The problem at order zero is governed by Orr Sommerfeld equation (Eq.(39)) associated with boundary conditions (Eqs.(41); (42); (43); (44)) in which we put $\alpha = 0$. Its solution writes for the stream function say $\psi^{(0)}$

$$\psi^{(0)}(y) = \eta(y+1)^2 \tag{47}$$

and for the celerity say $c^{(0)}$

$$c^{(0)} = 2 \tag{48}$$

This result is similar but not identical to the one obtained by Nsom et al. (2019) and Charru (2007). Indeed, the solution description is the same but the expression of the interface velocity U_0 is not the same.

At first order, the problem is governed by Orr Sommerfeld equation (Eq.(39)) associated with boundary conditions (Eqs.(41); (42); (43); (44)) in which we put $\alpha = 1$. Its solution writes for the stream function say $\psi^{(1)}$

$$D^4\psi^{(1)}(y) = iRe\left((U_0 - c^{(0)})D^2 - D^2U_0\right)\psi^{(0)} \tag{49}$$

Whose solution provides celerity at first order in the form

$$c^1 = iRe\frac{8}{15}\left(1 - \frac{5}{8}\left(\frac{1}{F_r} + \frac{\alpha^2}{We}\right)\right) \tag{50}$$

5.2 Marginal stability and parametric study

The marginal stability states correspond to $c_1 = 0$. As Reynolds number and Froude number are related by Equ.(46), the latter secular equation (Equ.(49)) involves two parameters say, Reynolds number Re and reduced wavenumber $\alpha^2/_{We}$. For assigned value of any of these parameters, Equ.(49) is solved by a shooting method. A trial value is given to one of them and the value of the other parameter is sought in order to satisfy Equ.(49).

So, for given slope, the marginal stability curve is obtained as follows: a value of reduced wavelength is fixed and growing Reynolds numbers are used for computing Equ.(49). The values obtained for celerity $c_1$are negative (defining stable flow) and then positive (defining unstable flow). Stable and unstable flows are separated by the marginal state where $c_1 = 0$ and the corresponding value of the Reynolds number defines the critical Reynolds number noted Re_c.

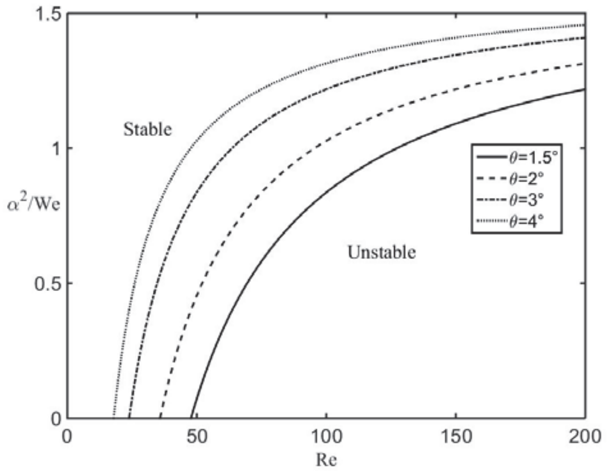

Figure 7. The marginal stability curve separating the stable states from the unstable ones for assigned flow configurations**.**

The computations brought out for gentle slopes, such that $1° \leq \beta \leq 6°$ (Allouche, 2015 ; Nsom et al., 2018) lead to the marginal stability curves presented in Fig.7. They show that when the slope increases, the critical Reynolds number decreases for given reduced wavelength. Moreover, as we can see in Fig.2, inertia acts along the x – axis while pressure (hydrostatic) acts along the y – axis. Therefore, inertia has a destabilizing effect while pressure has a stabilizing effect.

6. Discussion and Conclusions

The stability of a thin film propagating beneath a large quantity of ambient static non miscible lighter liquid and over a sloping plane was considered theoretically. Such configuration that has never been considered earlier can model the spill of a heavy hydrocarbon into the ocean by a tanker, following a voluntary or accidental degassing or an act of war. Equations of conservation of the mass and the momentum were appropriately made non dimensional and a similar solution was proposed in this paper. In this way, an analytical expression of the hydrodynamic field, say velocity field and pressure field is provided. Then, the equation governing the spatiotemporal evolution of the water-oil interface was built and solved by a perturbation method. Notably, three flow regimes were identified, say the inertial the viscous and the asymptotic regime in the height spatiotemporal evolution, for assigned aspect ratio λ **.** Also, the time evolution of the wave front position along the inclined plane was built.

Notably, an appropriate non dimensional form of the velocity field shows that it is similar to the case where the surrounding fluid is the atmosphere but with a different average velocity. Consequently, the stability analysis of both configurations and the results of both problems were similar. Notably, the solution to the secular equation showed that at zeroth order, there is no instability with respect to the long wave perturbations considered.

At first order, the secular equation was solved numerically by a shooting method. The marginal stability curve was built and the effect of the different forces acting on the flow has been pointed out. It was particularly shown that pressure and surface tension have a stabilizing effect, while inertia has a destabilizing effect.

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