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ARTICLE TEMPLATE

Improved Model Order-Reduction Techniques with Error-Bounds⋆

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ABSTRACT
This paper introduces two enhanced model order reduction techniques designed for scenarios involving frequency-weighted and frequency-limited-interval Gramians in the continuous-time domain. The primary objective is to address the instability issue identified in existing approaches in the continuous-time domain, as formulated by Enns for frequency-weighted scenarios and by Gawronski & Juang for frequency-limited-interval scenarios. Despite numerous solutions proposed in the literature to mitigate this problem, a persistent challenge remains the high approximation error between the original and reduced-order systems. To overcome this limitation, the proposed improved techniques focus on ensuring stability in reduced-order models while simultaneously minimizing the approximation error between the original and reduced systems. Furthermore, these enhanced techniques provide a computationally straightforward, *a priori* error-bound formula. Numerical findings underscore the correctness and efficiency of the proposed techniques in reducing the approximation error while maintaining stability, thereby substantiating their efficacy.

KEYWORDS
Reduced-order model; frequency limited-interval; frequency-weights; Gramians; error-bound.

1. Introduction

1.1. Motivation & Encouragement

Large systems like nuclear reactors, autonomous systems, sensor systems, filters, biomedical systems, chemical systems, data imaging, power networks, etc., become complex due to several system parameters and modeling equations. The intricate design, modeling, and analysis of this large-scale dynamical system are complex. Therefore, it is preferable to design or simulate a mathematical model before constructing the complete hardware of the dynamical systems. This practical approach is useful

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for analyzing some crucial parameters of the desired system, including input-output behavior, stability, passivity, etc. However, the problem arises when we have to design the mathematical model of complex, large dynamical systems (Cai & Chang, 2023; Du et al., 2021; Jin, Xiao, Song, & Qi, 2023; Li & Jiang, 2023). The reason behind this challenge is the need for more computational resources (Li & Jiang, 2022; Li, Jiang, & Mu, 2021). So, model order-reduction (MOR) is an algorithm used to extract reduced-order models (ROMs) from high order systems in order to facilitate ease in designing, optimization, simulation, and analysis of large-scale complex dynamical systems (Burohman, Besselink, Scherpen, & Camlibel, 2023; Ren & Wang, 2023). MOR captures the primal features of the original system while performing the reduction process (Abbasi, Mahmood, Khaliq, & Imran, 2022; Imran, 2022). The main task of MOR is to construct stable ROMs with less approximation error between the approximated and original mathematical model of the dynamical system (Imran, 2023; Imran & Imran, 2022a, 2022b).

1.2. Literature Review

In the MOR process, Balanced Truncation (BT) is the most common strategy to capitulate the low-order model from the high-order model. It not only ensures the stability of ROM but also provides error-bound. In BT, the least controllable and observable states are discarded, and the most significant observable and controllable states are used to consider the low-order approximation of the original system (Hamdani, Imran, & Imran, 2022; Schröder & Voigt, 2023). Generally, BT performs the reduction process by using the full frequency range to compute the system's response. However, in some applications like filter and controller reduction, it is preferred to consider frequency-response approximation error over a certain frequency band of interest as sometimes the reduction error (approximation error) is more significant in a particular range of frequency. It builds the concept of frequency weights in the MOR process (Ghafoor & Imran, 2017, 2021; Liang, Chen, He, & Chen, 2019; Toor et al., 2019). Enns technique (Enns, 1984) upgraded the BT strategy and presented a frequency-weighted model-reduction scenario. However, this technique only ensures the stability for single-sided frequency-weights (input or output), and it may not yield stable ROMs in case of both-sided frequency-weights (inputs and outputs) because of some input-output related symmetric matrices that are not conserved to be positive-definite or semi-positive definite. So, many frequency-weighted model-reduction techniques have been presented in literature to modify symmetric matrices in order to create stable ROMs for both-sided frequency-weights (Ghafoor & Sreeram, 2007; Imran, Ghafoor, & Sreeram, 2014; Lin & Chiu, 1990; Sreeram, 2004; Varga & Anderson, 2003; Wang, Sreeram, & Liu, 1999).

From the existing work, we have considered (Batool, Imran, Imran, & Ahmad, 2022; Imran et al., 2014; Wang et al., 1999) for comparison with our proposed work; however, these proposed techniques eventuate in high approximation error between original and ROMs with complex error-bound for some dynamic systems. *Wang et al.* (Wang et al., 1999) suggested absolute function to modify symmetric matrices for developing stable ROMs. However, *Wang et al.* (Wang et al., 1999) method eventuates in high approximation error between original and ROM with complex error-bound for some dynamic systems because of large variations in some eigenvalues of original systems as it does not affect all the eigenvalues equally. *Imran et al.* (Imran et al., 2014) introduced a method to subtract the least negative eigenvalue from all eigenvalues to ensure positive-definite or semi-definiteness of symmetric matrices for creating stable ROMs, but it leads to large approximation error due to nullification of the last eigenvalue. Later, *Sammana et al.* (Batool et al., 2022) achieved stability by multiplying consecutive negative-eigenvalues to make symmetric matrices positive-definite or semi-positive definite. However, *Sammana et al.* (Batool et al., 2022) method also creates a large approximation error between original and ROM with complex error-bound because it does not affect all the eigenvalues equally. *Gawronski et al.* (Gawronski & Juang, 1990) presented the concept of frequency limited-interval scenario in MOR procedure that simplified the MOR algorithm and considered the approximation in required frequency-range instead of constructing frequency-weights. So, for this technique, the controllability and observability of Gramians are defined for frequency limited interval. However, the frequency limited-interval also capitulates unstable ROMs like frequency-weighted scenario for original stable system because input-output related symmetric matrices are not conserved to be positive-definite or semi-positive definite. Moreover, it does not compute error bound. So, to solve the stability issue in frequency limited-interval scenario *Gugercin et al.* (Gugercin & Antoulas, 2004) used absolute function of negative-eigenvalues to en-sure the positive or semi-positive definiteness of symmetric matrices to achieve stable ROMS, and *Ghafoor et al.*(Ghafoor & Sreeram, 2008) developed stable ROMs by truncating all negative-eigenvalues and retaining only positive eigenvalues. But (Ghafoor & Sreeram, 2008; Gugercin & Antoulas, 2004) techniques do not affect all negative-eigenvalues equally, which lead to large approximation error in some systems. *Imran et al.*. (Ghafoor & Imran, 2015) guaranteed the stable ROMs in the desired frequency range by applying the frequency-weighted technique (Imran et al., 2014) in frequency limited-interval scenario. Later, *Sammana et al.* (Batool et al., 2022) introduced an algorithm that combines four different operations to create stable ROMs: power of total negative-eigenvalues, inverse power of total negative-eigenvalues, absolute and addition of consecutive eigenvalues. *Sammana et al.* (Batool et al., 2022) algorithm increases the overall computational complexity of the method that eventuates in large approximation error with complex error-bound. So, stability is achieved by existing techniques (Batool et al., 2022; Ghafoor & Imran, 2015; Ghafoor & Sreeram, 2008; Gugercin & Antoulas, 2004) but at the cost of large approximation error and complex error-bound.

1.3. Main Contribution & Paper Organization

In this research, the development of effective MOR techniques for linear time-invariant (LTI) systems has been a continual pursuit in the control systems community. Addressing the limitations of existing approaches, our work introduces two innovative MOR methods specifically designed for frequency-weighted and frequency-limited interval scenarios in the continuous-time domain. The primary motivation is to offer stability assurances for ROMs by ensuring the positive-definiteness or semi-positive definiteness of symmetric matrices. These methods present a departure from traditional techniques by providing equal impact on all negative eigenvalues, avoiding eigenvalue truncation, and simplifying error-bound computations. The subsequent description highlights the distinctive features of our proposed techniques.

- Two novel and enhanced MOR techniques are tailored for frequency-weighted and frequency-limited interval scenarios in LTI systems within the continuous time domain.

- Assurance of positive-definiteness or semi-positive definiteness of symmetric matrices, ensuring the stability of ROMs in the proposed methods.
- Equal impact on all negative eigenvalues without any eigenvalue truncation, distinguishing the proposed methods from previous approaches (Ghafoor & Imran, 2015; Ghafoor & Sreeram, 2007, 2008; Gugercin & Antoulas, 2004; Imran et al., 2014; Wang et al., 1999).
- Simplified error-bound computation is in contrast to (Batool, Imran, ELAHI, MAQBOOL, & GILANI, 2021; Batool et al., 2022), as the proposed techniques offer a computationally straightforward approach with a more easily calculable frequency response *a priori* error-bound formula for both frequency-weighted and frequency-limited interval scenarios.

Numerical results and simulations are presented and compared with existing stability-preserving MOR techniques (Batool et al., 2022; Enns, 1984; Imran et al., 2014; Wang et al., 1999) for frequency-weighted scenario and (Batool et al., 2021; Gawronski & Juang, 1990; Ghafoor & Imran, 2015; Ghafoor & Sreeram, 2007; Gugercin & Antoulas, 2004) for frequency limited-interval scenario.
Furthermore, Section 2 provides a theoretical foundation, exploring the intricacies of frequency-weighted and frequency-limited interval scenarios. In Section 3, we detail our main result, ensuring positive-definiteness or semi-positive definiteness of symmetric matrices, overcoming stability challenges in ROMs, and surpassing the limitations of current methods. Section 4 shifts to practical validation through numerical simulations, substantiating the computational simplicity and superior performance of our proposed techniques compared to existing methods. The paper concludes in Section 5, summarizing key findings and charting future research directions in MOR for LTI systems.

2. Preliminaries

Consider the following transfer function of a LTI continuous-time system be given as:

$$G_o(s) = C_o(sI - A_o)^{-1}B_o + D_o \, (1)$$

where $\{A_o \in R^{n\times n},\ B_o \in R^{n\times m},\ C_o \in R^{p\times n},\ D_o \in R^{p\times m}\}$ n, m and p represent original system order, number of inputs and outputs of the system respectively. The main task of MOR is to find as:

$$G_z(s) = C_z(sI - A_z)^{-1}B_z + D_z \tag{2}$$

which is the approximated model of original system using full frequency-range, where $\{A_z \in R^{z\times z}, B_z \in R^{z\times m},\ C_z \in R^{p\times z},\ D_z \in R^{p\times m}\}$ and z is the order of ROM. Let P_{c_o} and Q_{b_O} are the controllability and observability Gramians respectively, which are presented below as:

$$P_{c_o} = \int_{-\infty}^{\infty} e^{A_o t} B_o B_o^T e^{A_o{}^T t} dt \tag{3}$$

$$Q_{b_o} = \int_{-\infty}^{\infty} e^{A_o^T t} C_o^T C_o e^{A_o t} dt \tag{4}$$

The above defined Gramians are the solution of following Lyapunov equations:

$$A_o P_{c_o} + P_{c_o} A_o^T + B_o B_o^T = 0 \tag{5}$$

$$A_o^T Q_{b_o} + Q_{b_o} A_o + C_o^T C_o = 0 \tag{6}$$

A non-singular transformation matrix T is used to obtain a balanced system from dynamic system by converting observability and controllability Gramians into equal and diagonal matrices:

$$T^T Q_{b_o} T = T^{-1} P_{c_o} T^{-T} = \Sigma = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \tag{7}$$

where $\sigma_s \geq \sigma_{s+1}$, $s = 1, 2, 3, \ldots, n-1$, $\sigma_z > \sigma_{z+1}$ and formulate the Hankel Singular Values (HSV) of Σ that are used to measure the robustness of the observable and controllable state (Imran et al., 2014). The ROM is constructed by applying the following transformation over the original system as:

$$\hat{A}_t = T^{-1} A_o T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \hat{B}_t = T^{-1} B_o = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{8}$$

$$\hat{C}_t = C_o T = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad \hat{D}_t = D_o \tag{9}$$

where $A_{11} \in R^{z \times z}$. The ROM is obtained as: $G_z(s) = C_1(sI - A_{11})^{-1} B_1 + D_o$.

2.1. *Frequency-Weighted Model-Reduction Scenario*

Consider the transfer function of the stable input-weights for continuous-time system be given as (Enns, 1984):

$$V_u(s) = C_u(sI - A_u)^{-1} B_u + D_u \tag{10}$$

where $\{A_u \in R^{n_u \times n_u}, B_u \in R^{n_u \times m_u}, C_u \in R^{p_u \times n_u}, D_u \in R^{p_u \times m_u}\}$ and transfer function of the stable output-weights is presented as:

$$W_q(s) = C_q(sI - A_q)^{-1} B_q + D_q \tag{11}$$

where $\{A_q \in R^{n_q \times n_q}, B_q \in R^{n_q \times m_q}, C_q \in R^{p_q \times n_q}, D_q \in R^{p_q \times m_q}\}$. The augmented systems are created below

$$G_o(s) V_u(s) = C_x(sI - A_x)^{-1} B_x + D_x \tag{12}$$

$$W_q(s) G_o(s) = C_y(sI - A_y)^{-1} B_y + D_y \tag{13}$$

where

$$\left[\begin{array}{c|c} A_x & B_x \\ \hline C_x & D_x \end{array}\right] = \left[\begin{array}{cc|c} A_o & B_o C_u & B_o D_u \\ 0 & A_u & B_u \\ \hline C_o & D_o C_u & D_o D_u \end{array}\right], \quad \left[\begin{array}{c|c} A_y & B_y \\ \hline C_y & D_y \end{array}\right] = \left[\begin{array}{cc|c} A_q & B_q C_o & B_q D_o \\ 0 & A_o & B_o \\ \hline C_q & D_q C_o & D_q D_o \end{array}\right]$$

Let the Gramians be defined as:

$$P_x = \begin{bmatrix} P_e & P_{12} \\ P_{12}^T & P_v \end{bmatrix}, \quad Q_y = \begin{bmatrix} Q_w & Q_{12}^T \\ Q_{12} & Q_e \end{bmatrix} \tag{14}$$

satisfying the following Lyapunov equations:

$$A_x P_x + P_x A_x^T + B_x B_x^T = 0 \tag{15}$$

$$A_y^T Q_y + Q_y A_y + C_y^T C_y = 0 \tag{16}$$

By expanding the 1^{st} and 4^{th} block of the equations (15) and (16) respectively, compute

$$A_o P_e + P_e {A_o}^T + X_w = 0 \tag{17}$$

$${A_o}^T Q_e + Q_e A_o + Y_w = 0 \tag{18}$$

where

$$X_w = B_o C_u P_{12}^T + P_{12} C_u^T B_o^T + B_o D_u D_u^T B_o^T \tag{19}$$

$$Y_w = C_o^T B_q^T Q_{12}^T + Q_{12} B_q C_o + C_o^T D_q^T D_q C_o \tag{20}$$

By eigenvalue-decomposition of X_w and Y_w we have the following

$$X_w = U_w S_w U_w^T = U_w diag[S_{w_1}, S_{w_2}] U_w^T \tag{21}$$

$$B_w = U_w S_w^{1/2} = U_w diag[S_{w_1}^{1/2}, S_{w_2}^{1/2}] \tag{22}$$

$$Y_w = V_w R_w V_w^T = V_w diag[R_{w_1}, R_{w_2}] V_w^T \tag{23}$$

$$C_w = R_w^{1/2} V_w^T = diag[R_{w_1}^{1/2}, R_{w_2}^{1/2}] V_w^T \tag{24}$$

where

$$S_{w_1} = diag(s_1, s_2 \ldots., s_j), \; S_{w_2} = diag(s_{j+1}, s_{j+2} \ldots\ldots, s_n)$$

$$R_{w_1} = diag(r_1, r_2 \ldots\ldots.., r_t), \; R_{w_2} = diag(r_{t+1}, r_{t+2} \ldots.., r_n)$$

where j and t represent number of positive eigenvalues of X_w and Y_w, respectively. Let the contragredient matrix T be obtained as:

$$T^T Q_y T = T^{-1} P_x T^{-T} = \Sigma \tag{25}$$

where Σ formulates the HSV which are arranged in the descending order. By computing transformation and partitioning the original system, the ROMs are obtained in a similar way as in equations (8) and (9).

Remark 1. Since in *Enns* technique (Enns, 1984), $X_w \geq 0$ and $Y_w \geq 0$ are not always guaranteed. So, in case of double sided frequency-weights the ROM may not remain stable (Wang et al., 1999).

2.2. Frequency-Limited Model-Reduction Scenario

Let controllability P_{G_j} and observability Q_{G_j} Gramians are defined for frequency limited-interval as $P_{G_j} = P(\omega_2) - P(\omega_1)$ and $Q_{G_j} = Q(\omega_2) - Q(\omega_1)$ respectively. The Gramians are expressed using Parseval's relationship as:

$$P_{G_j} = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} (j\omega I - A_o)^{-1} B_o B_o^T (-j\omega I - A_o^T)^{-1} d\omega \tag{26}$$

$$Q_{G_j} = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} (-j\omega I - A_o^T)^{-1} C_o^T C_o (j\omega I - A_o)^{-1} d\omega \tag{27}$$

These Gramians satisfy the following Lyapunov equations:

$$A_o P_{G_j} + P_{G_j} A_o^T + X_g = 0 \tag{28}$$
$$A_o^T Q_{G_j} + Q_{G_j} A_o + Y_g = 0 \tag{29}$$

where

$$\begin{aligned}
X_g &= (S_o(\omega_2) - S_o(\omega_1)) B_o B_o{}^T + B_o B_o^T (S_o^*(\omega_2) - S_o^*(\omega_1)) \\
Y_g &= (S_o^*(\omega_2) - S_o^*(\omega_1)) C_o^T C_o + C_o^T C_o (S_o(\omega_2) - S_o(\omega_1)) \\
S_o(\omega) &= \frac{j}{2\pi} \ln((j\omega I + A_o)(-j\omega I + A_o)^{-1})
\end{aligned}$$

Eigenvalue-decomposition of X_g and Y_g yield

$$X_g = U \begin{bmatrix} S_{j_1} & 0 \\ 0 & S_{j_2} \end{bmatrix} U^T, Y_g = V \begin{bmatrix} R_{j_1} & 0 \\ 0 & R_{j_2} \end{bmatrix} V^T \tag{30}$$

$$S_{j_1} = diag(s_1, ..., s_j) \geq 0, S_{j_2} = diag(s_{j+1}, ..., s_n) < 0$$
$$R_{j_1} = diag(r_1, ..., r_t) \geq 0, R_{j_2} = diag(r_{t+1}, ..., r_n) < 0$$

$j \leq n$ and $t \leq n$ represent positive eigenvalues of X_g and Y_g matrices respectively. $S_o^*(\omega)$ is the conjugate transpose of $S_o(\omega)$. The contragredient matrix T_j is obtained as:

$$T_j^T Q_{G_j} T_j = T_j^{-1} P_{G_j} T_j^{-T} = \Sigma \tag{31}$$

The ROM $G_z(s) = C_z(sI - A_z)^{-1} B_z + D$ is derived after computing the transformation and partitioning the original system in a similar way as in equations (8) and (9).

Remark 2. In *Gawronski et al.* technique (Gawronski & Juang, 1990), the symmetric matrices $X_g \geq 0$ and $Y_g \geq 0$ are not always guaranteed that lead to yield unstable ROMs (Gugercin & Antoulas, 2004).

3. Main Results

Although the primary frequency-weighted and frequency limited-interval MOR scenarios for continuous-time systems proposed by Enns (Enns, 1984) and *Gawronski et al.* (Gawronski & Juang, 1990) respectively compute lowest approximation error, at the cost of unstable ROMs. The reason behind this instability issue is the symmetric matrices $X_Z \in \{X_w, X_g\}$, $Y_Z \in \{Y_w, Y_g\}$ which are not conserved to be positive or semi-positive definite. So, we have applied the two proposed improved techniques on frequency-weighted and frequency limited-interval Gramians-based model-reduction scenarios to compute approximation error with easily calculable *a priori* error bound in comparison with existing stability conserving techniques. In first technique, the negative eigenvalue is subtracted from the sine function of negative eigenvalue to produce stable ROMs. In second technique, we have used exponential of inverse of negative eigenvalue and order of system-matrix (A_o) as power function (n) to compute stable ROMs. The proposed improved techniques yield stable ROMs with low approximation error by building some variations in the matrices $\{X_Z, Y_Z\}$ to ensure the positive/semi-positive definiteness of input/output related symmetric matrices respectively. Moreover, we have presented numerical simulations at different Reduced-Orders (ROs) to show the efficacy of both techniques. The response of both techniques is different at same ROM of desired system.

3.1. Proposed Techniques

Let a new controllability P_{Z_k} and observability Q_{Z_k} Gramians respectively, are calculated by solving the following Lyapunov equations:

$$A_o P_{Z_k} + P_{Z_k} A_o^T + B_{Z_k} B_{Z_k}^T = 0 \tag{32}$$

$$A_o^T Q_{Z_k} + Q_{Z_k} A_o + C_{Z_k}^T C_{Z_k} = 0 \tag{33}$$

where $k = 1, 2$. For indefinite symmetric matrices X_Z and Y_Z the new input, output related matrices are defined as B_{Z_k} and C_{Z_k} respectively:

$$B_{Z_1} = \begin{cases} U_{Z_1} S_{Z_1}^{1/2} \\ U_{Z_2}(sin(S_{Z_2}) - S_{Z_2})^{1/2} \end{cases} \tag{34}$$

$$B_{Z_2} = \begin{cases} U_{Z_1} S_{Z_1}^{1/2} \\ U_{Z_2}((exp(1/S_{Z_2}))^n)^{1/2} \end{cases} \tag{35}$$

$$C_{Z_1} = \begin{cases} R_{Z_1}^{1/2} V_{Z_1}^T \\ (sin(R_{Z_2}) - R_{Z_2})^{1/2} V_{Z_2}^T \end{cases} \tag{36}$$

$$C_{Z_2} = \begin{cases} R_{Z_1}^{1/2} V_{Z_1}^T \\ ((exp(1/R_{Z_2}))^n)^{1/2} V_{Z_2}^T \end{cases} \tag{37}$$

where n is the order of the system matrix A_o and the terms U_{Z_1}, U_{Z_2}, S_{Z_1}, S_{Z_2}, V_{Z_1}, V_{Z_2}, R_{Z_1} and R_{Z_2} are attained from the eigenvalue-decomposition of symmetric matrices,

$$X_Z = [U_Z\, S_Z\, U_Z^T] = [U_{Z_1}\;\; U_{Z_2}] \begin{bmatrix} S_{Z_1} & 0 \\ 0 & S_{Z_2} \end{bmatrix} \begin{bmatrix} U_{Z_1}^T \\ U_{Z_2}^T \end{bmatrix}$$
$$Y_Z = [V_z\, R_Z\, V_Z^T] = [V_{Z_1}\;\; V_{Z_2}] \begin{bmatrix} R_{Z_1} & 0 \\ 0 & R_{Z_2} \end{bmatrix} \begin{bmatrix} V_{Z_1}^T \\ V_{Z_2}^T \end{bmatrix}$$
$$S_{Z_1} = diag(s_1, ..., s_j), S_{Z_2} = diag(s_{j+1}, s_{j+2}....., s_n),$$
$$R_{Z_1} = diag(r_1, ..., r_t), R_{Z_2} = diag(r_{t+1}, r_{t+2}....., r_n).$$

where j and t are the indexes of the positive eigenvalues.

Remark 3. When $X_Z > 0$ and $Y_Z > 0$, $B_{Z_k} = S_{Z_k} S_{Z_k}^{1/2}$ and $C_{Z_k} = R_{Z_k}^{1/2} V_{Z_k}^T$.

Let a contragradient transformation matrix T_{Z_k} is derived as

$$T_{Z_k}^T Q_{Z_k} T_{Z_k} = T_{Z_k}^{-1} P_{Z_k} T_{Z_k}^{-T} = diag(\sigma_1, \sigma_2 \cdots \sigma_n) \tag{38}$$

where $\sigma_s \geq \sigma_{s+1}$, $s = 1, 2, 3, ..., n-1$, $\sigma_z > \sigma_{z+1}$. A ROM $G_z(s) = C_1(sI - A_{11})^{-1} B_1$ is attained by applying the following transformation

$$T_{Z_k}^{-1} A_o T_{Z_k} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad T_{Z_k}^{-1} B_o = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$
$$C_o T_{Z_k} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \qquad D_{Z_k} = D_o$$

Remark 4. The incorporation of non-linear terms (i.e., sin(.) and exp(.)) in B_{Z_k} and C_{Z_k} serves a precise and controlled purpose in our stability preservation strategies for LTI systems. These non-linear elements are strategically introduced to address the instability associated with indefinite matrices, specifically in X_Z and Y_Z . By utilizing these non-linear terms, our approach ensures that each element of X_Z and Y_Z becomes positive or semi-positive definite, thereby achieving stability in ROMs. This strategic incorporation serves as a transformative measure, effectively converting these matrices into positive or semi-positive definite forms, facilitating the attainment of a stable and linear system representation.

Remark 5. Since $X_Z \leq B_{Z_k} B_{Z_k}^T$, $Y_Z \leq C_{Z_k}^T C_{Z_k}$, $P_{Z_k} > 0$ and $Q_{Z_k} > 0$. Hence, the realization (A_o, B_{Z_k}, C_{Z_k}) is minimal and stable ROM is guaranteed.

3.2. Error Bounds

Theorem 3.1. *The following error bound holds for the proposed improved model-reduction techniques for frequency-weighted scenario,*

$$||W_q(s)(G_o(s) - G_z(s))V_u(s)\|_\infty \leq 2\|W_q(s)L_{Z_k}\|_\infty \|K_{Z_k} V_u(s)\|_\infty \sum_{s=z+1}^{n} \sigma_s$$

if the rank conditions rank $[B_{Z_k} B_o] = rank\,[B_{Z_k}]$ *and rank* $\begin{bmatrix} C_{Z_k} \\ C_o \end{bmatrix} = rank\,[C_{Z_k}]$ *are satisfied, where*

$$L_{Z_1} = \begin{cases} CV_{Z_1} R_{Z_1}^{-1/2} \\ CV_{Z_2}(sin(R_{Z_2}) - R_{Z_2})^{-1/2} \end{cases} \tag{39}$$

$$L_{Z_2} = \begin{cases} CV_{Z_1} R_{Z_1}^{-1/2} \\ CV_{Z_2}((exp(1/R_{Z_2})^n)^{-1/2} \end{cases} \tag{40}$$

$$K_{Z_1} = \begin{cases} S_{Z_1}^{-1/2} U_{Z_1}^T B \\ (sin(S_{Z_2}) - (S_{Z_2})^{-1/2} U_{Z_2}^T B \end{cases} \tag{41}$$

$$K_{Z_2} = \begin{cases} S_{Z_1}^{-1/2} U_{Z_1}^T B \\ ((exp(1/S_{Z_2})^n)^{-1/2} U_{Z_2}^T B \end{cases} \tag{42}$$

Proof. As the $rank\,[B_{Z_k}\ B_o]$=$rank\,[B_{Z_k}]$ and $rank \begin{bmatrix} C_{Z_k} \\ C_o \end{bmatrix} = rank\,[C_{Z_k}]$ holds. By substituting $B_1 = B_{Z_1} K_{Z_k}$, $C_1 = L_{Z_k} C_{Z_1}$, $B = B_{Z_k} K_{Z_k}$ and $C = L_{Z_k} C_{Z_k}$, respectively, computes:

$$\begin{aligned} \|W_q(s)(G_o(s) - G_z(s))V_u(s)\|_\infty &= \|W_q(s)(C_o(sI - A_o)^{-1} B_o - C_1(sI - A_{11})^{-1} B_1)V_u(s)\|_\infty \\ &= \|W_q(s)(L_{Z_k} C_{Z_k}(sI - A_o)^{-1} B_{Z_k} K_{Z_k} - L_{Z_k} C_{Z_1}(sI - A_{11})^{-1} B_{Z_1} K_{Z_k})V_u(s)\|_\infty \\ &= \|W_q(s) L_{Z_k}(C_{Z_k}(sI - A_o)^{-1} B_{Z_k} - C_{Z_1}(sI - A_{11})^{-1} B_{Z_1}) K_{Z_k} V_u(s)\|_\infty \\ &\leq \|W_q(s) L_{Z_k}\|_\infty \|(C_{Z_k}(sI - A_o)^{-1} B_{Z_k} - C_{Z_1}(sI - A_{11})^{-1} B_{Z_1})\|_\infty \|K_{Z_k} V_u(s)\|_\infty \end{aligned}$$

If $\{A_{11}, B_{Z_1}, C_{Z_1}, D_o\}$ is the ROM, which is attained by splitting a balanced realization $\{A_o, B_{Z_k}, C_{Z_k}, D_o\}$, we have from (Wang et al., 1999), (Imran et al., 2014), $\|(C_{Z_k}(sI - A_o)^{-1} B_{Z_k} - C_{Z_1}(sI - A_{11})^{-1} B_{Z_1})\|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_s$. Therefore,

$$\|W_q(s)(G_o(s) - G_z(s))V_u(s)\|_\infty \leq 2\|W_q(s) L_{Z_k}\|_\infty \|K_{Z_k} V_i(s)\|_\infty \sum_{s=z+1}^{n} \sigma_s$$

□

Remark 6. When input frequency-weight $V_u(s)$ and output frequency-weight $W_q(s)$ become unity then the error bound expression reduced to $\|(G_o(s) - G_z(s))\|_\infty \leq 2\|L_{Z_k}\|_\infty \|K_{Z_k}\|_\infty \sum_{s=z+1}^{n} \sigma_s$; consequently, expression holds for frequency-limited model-reduction scenario.

Remark 7. Two choices of $K_{Z_k} \in \{K_{Z_1}; K_{Z_2}\}$ and $L_{Z_k} \in \{L_{Z_1}; L_{Z_2}\}$ form basis to derive error bounds for each proposed technique.

3.3. Algorithmic Framework

To facilitate clarity and comprehension, the algorithmic framework is described in Algorithm 1. The proposed Algorithms 1 aim to achieve stable and linear ROMs for LTI systems in the continuous-time domain. The process involves integrating nonlin-ear treatments strategically to address challenges associated with indefinite matrices, particularly in the cases of B_o and C_o. The algorithms focus on transforming these matrices into positive or semi-positive definite forms, ensuring stability in the ROMs while preserving the inherent linearity crucial for LTI system realization.

Algorithm 1 Computation of ROM

procedure COMPUTEROM($G_o(s), V_u(s), W_q(s), \omega_1, \omega_2$)
    Inputs:
        $G_o(s)$ ▷ Original transfer function
        $V_u(s), W_q(s)$ ▷ Weights
        ω_1, ω_2 ▷ Frequency interval
    Output:
        $G_z(s)$ ▷ Reduced Order Model
    Main Steps:
    for each frequency ω in $[\omega_1, \omega_2]$ **do**
        $X_Z, X_g, Y_Z, Y_g \leftarrow$ PERFORMEIGENVALUEDECOMPOSITION($G_o(s), V_u(s), W_q(s), \omega$)
        $B_{Z_k}, C_{Z_k} \leftarrow$ COMPUTEBCMATRICES(X_Z, X_g, Y_Z, Y_g)
        $P_{Z_k}, Q_{Z_k} \leftarrow$ SOLVEEQUATIONS(32, 33)
        $T_{Z_k} \leftarrow$ SOLVETRANSFORMATIONMATRIX(38)
        $G_z(s) \leftarrow$ PARTITIONBALANCEDREALIZATION($T_{Z_k}, B_{Z_k}, C_{Z_k}$)
        STOREROM($G_z(s)$)
    end for
end procedure

function PERFORMEIGENVALUEDECOMPOSITION($G_o(s), V_u(s), W_q(s), \omega$)
    return X_Z, X_g, Y_Z, Y_g
end function

function COMPUTEBCMATRICES(X_Z, X_g, Y_Z, Y_g)
    return B_{Z_k}, C_{Z_k}
end function

function SOLVEEQUATIONS(equations)
    return P_{Z_k}, Q_{Z_k}
end function

function SOLVETRANSFORMATIONMATRIX(equation (38))
    return T_{Z_k}
end function

function PARTITIONBALANCEDREALIZATION($T_{Z_k}, B_{Z_k}, C_{Z_k}$)
    return $G_z(s)$
end function

procedure STOREROM($G_z(s)$)
    STORE($G_z(s)$)
end procedure

4. Numerical Simulations

In this section, we showcase the effectiveness of the proposed improved model-reduction techniques through simulations of high-order systems. The numerical examples presented here serve as a comprehensive illustration of the performance of our techniques, emphasizing their practical applicability.
The following simulations demonstrate the ability of our techniques to handle high-order systems while maintaining a high level of accuracy. To quantify the performance, we have considered numeric values representing the approximation error up to five digits precision, providing a nuanced understanding of the proposed techniques' efficacy.

4.1. Frequency-Weighted Model-Reduction Scenario

Example 1: Consider a 6^{th} order original stable system (Imran et al., 2014) with following stable input-output frequency weights

$$A_i = \begin{bmatrix} -2.25 & -0 \\ 0 & -0.05 \end{bmatrix}, B_i = \begin{bmatrix} 2 & 0.5 \\ 0.2 & 0.3 \end{bmatrix} C_i = \begin{bmatrix} 1.3 & 0.5 \\ 0.1 & 0.1 \end{bmatrix}, D_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_o = \begin{bmatrix} -4.2 & 0 \\ 0 & -0.025 \end{bmatrix}, B_o = \begin{bmatrix} 1 & 0.5 \\ 0.2 & 0.3 \end{bmatrix} C_o = \begin{bmatrix} 1.3 & 0.5 \\ 0.4 & 0.7 \end{bmatrix}, D_o = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The comparison of the approximation error and error bounds for different ROs is

Table 1. Error and Error Bounds Comparison for Example 1

RO	Enns (1984)	Wang et al. (1999)		Imran et al. (2014)		(Batool et al., 2022)		Proposed-I		Proposed-II	
	Error	Error	Bound	Error	Bound	Error	Bound	Error	Bound	Error	Bound
1	unstable	2462.8	6718.1	2724.4	30109	2604.3	3.7928e+05	2362.6	35378	2028.4	93130
2	12.742	136.68	2662.5	123.14	6768	137.81	1.492e+05	136.46	14152	129.69	41295
3	5.9973	39.9	718.24	225.28	3733.8	40.849	54607	36.545	3248.7	16.501	8103
4	0.24849	16.811	287.91	141.42	1745.5	22.858	19184	11.034	1317	4.89	3241.6
5	.079136	5.8769	68.684	9.417	415.71	13.678	4398.3	3.7963	302.23	1.7825	765.42

shown in Table 1. It can be noticed from Table 1 that the 1^{st} order model built by Enns technique (Enns, 1984) gives unstable ROM as the pole is located at $s = 0.0164$; whereas, the *Wang et al.* (Wang et al., 1999), *Imran et al.*, (Imran et al., 2014), *Sammana et al.* (Batool et al., 2022) and the proposed methods build stable ROMs. The proposed improved techniques capitulate low approximation error as compared to existing stability-conserving techniques.

Example 2: Consider a hospital building model sparse system with 48^{th} states (Chahlaoui & Van Dooren, 2002). The reduction of the given model is performed at different ROs to describe the efficacy of the proposed techniques. The model-reduction is done using the following stable input/output weights respectively,

$$V_u(s) = (0.6)/(s + 0.6), W_q(s) = (s + 3.201)/(s + 0.0006)$$

Table 2 illustrates the corresponding results of the given model at ROs where the approximation error is significant, and it can be noticed from the given table that the stable ROMs built by the proposed techniques capitulate low approximation error in comparison with existing stability conserving techniques.

Table 2. Error and Error Bounds Comparison for Example 2

RO	Enns (1984)	Wang et al. (1999)		Imran et al. (2014)		Sammana et al. (2022)		Proposed-I		Proposed-II	
	Error	Error	Bound	Error	Bound	Error	Bound	Error	Bound	Error	Bound
5	0.0009	22.81	218.3	1.411	1365	6317.8	24302	0.143	3928	0.063	55.45
7	$7.681e^{-}5$	7.896	134.4	9.597	1228	590.3	19235	5.257	2409	5.187	34.01
11	0.0001	15.57	60.59	2.398	9839.9	8.3519	11698	2.241	1085	2.234	15.31
21	$1.363e^{-}5$	1.020	4.985	14.92	5299.3	2.5243	37287	0.635	890.2	0.631	1.256
22	$2.42e^{-}6$	0.045	3.450	0.255	4926.8	0.17607	3249	0.028	617.2	0.028	0.871
23	$8.88e^{-}6$	0.177	2.443	4.206	4559.4	0.27169	28168	0.076	436.8	0.0760	0.616
25	$3.46e^{-}6$	0.215	1.017	2.954	3866.1	1.3695	2076	0.084	182.1	0.084	0.257
26	$2.015e^{-}7$	0.0006	0.603	0.031	3550.2	0.040544	17711	0.0004	107.7	0.0003	0.152
28	$1.16e^{-}7$	0.001	0.324	0.015	2997.3	0.045317	12535	0.0007	58.09	0.0007	0.081
31	$1.10e^{-}7$	0.017	0.160	25.08	2267.3	0.27742	73492	0.009	28.51	0.009	0.040
32	$4.03e^{-}8$	0.001	0.125	0.976	2044.9	0.038688	58817	0.0006	22.29	0.0006	0.031
33	$1.76e^{-}7$	0.005	0.100	20.21	1830.8	73.807	48355	0.002	17.87	0.002	0.025
34	$2.33e^{-}8$	0.001	0.075	0.013	1618.3	0.067925	38027	0.0006	13.48	0.0006	0.019
35	$4.15e^{-}8$	0.007	0.061	7.754	1429.9	0.15128	29784	0.002	11.06	0.002	0.015
36	$1.38e^{-}8$	0.0003	0.048	0.3655	1248.7	0.11549	21578	0.0001	8.682	0.0001	0.0122
38	$2.01e^{-}8$	0.0002	0.025	0.004	902.4	0.0038687	74719	0.0001	4.610	0.0001	0.0065
39	$5.38e^{-}8$	0.0049	0.015	0.585	759.1	0.27275	47582	0.0021	2.728	0.0021	0.0038

4.2. Frequency-Limited Model-Reduction Scenario

Example 3: Consider an example of a high-pass stable Chebyshev type-2 filter (Toor, Imran, Ghafoor, Zeeshan, & Rauf, 2018) of 8^{th} order with desired frequency-interval $[\omega_1, \omega_2] = [5, 11] rad/s$. The comparison of the error function singular values $\sigma[G_8(s) - G_1(s)]$ is shown in Fig. 1, where $G_8(s)$ is original order stable system and

$G_1(s)$ is the 1^{st} order ROM that is derived using (Batool et al., 2021; Ghafoor & Imran, 2015; Ghafoor & Sreeram, 2008; Gugercin & Antoulas, 2004) and proposed techniques respectively. In the desired frequency range $[\omega_1, \omega_2] = [5, 11]rad/s$, a close-up view of the error plot is illustrated in Fig. 2 to show the efficacy of the results of the proposed technique. It can be noted that the proposed techniques capitulate compara-ble approximation error as compared to *Gawronski et al.* (Gawronski & Juang, 1990) technique and compute low approximation error as compared to stability preservation techniques (Batool et al., 2021; Ghafoor & Imran, 2015; Ghafoor & Sreeram, 2008) within the desired frequency range.

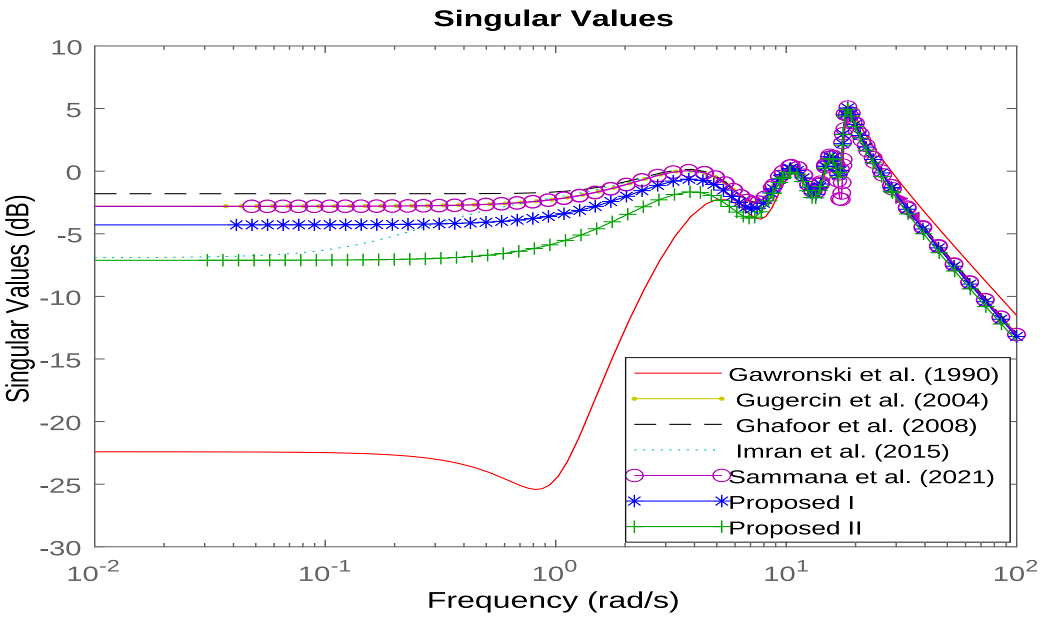

Figure 1. Frequency reponse error comparison $\sigma[G_8(s) - G_1(s)]$

Example 4: Consider an example of a high-pass stable Chebyshev type-2 filter (Toor et al., 2018) of 50^{th} order with desired frequency-interval $[\omega_1, \omega_2] = [9, 20]rad/s$. The comparison of the error function singular values $\sigma[G_{50}(s) - G_3(s)]$ is shown in Fig. 3, where $G_{50}(s)$ is original order stable system and $G_3(s)$ is the 3^{rd} order ROM that is derived using (Batool et al., 2021; Ghafoor & Imran, 2015; Ghafoor & Sreeram, 2008; Gugercin & Antoulas, 2004) and proposed techniques respectively. In the de-sired frequency range $[\omega_1, \omega_2] = [9, 20]rad/s$, a close-up view of the error plot is illustrated in Fig. 4 to show the efficacy of the results of the proposed technique. It can be noted that the proposed techniques capitulate comparable approximation error as compared to *Gawronski et al.* (Gawronski & Juang, 1990) technique and compute low approximation error as compared to stability preservation techniques (Batool et al., 2021; Ghafoor & Imran, 2015; Ghafoor & Sreeram, 2008) within the desired frequency range.

Example 5: Consider an example of a high-pass stable Chebyshev type-2 filter (Toor et al., 2018) of 20^{th} order with desired frequency-interval $[\omega_1, \omega_2] = [1, 9]rad/s$. The comparison of the error function singular values $\sigma[G_{20}(s) - G_5(s)]$ is shown in Fig. 5, where $G_{20}(s)$ is original order stable system and $G_5(s)$ is the 5^{th} order ROM that is derived using (Batool et al., 2021; Ghafoor & Imran, 2015; Ghafoor & Sreeram, 2008; Gugercin & Antoulas, 2004) and proposed techniques respectively. In the de-

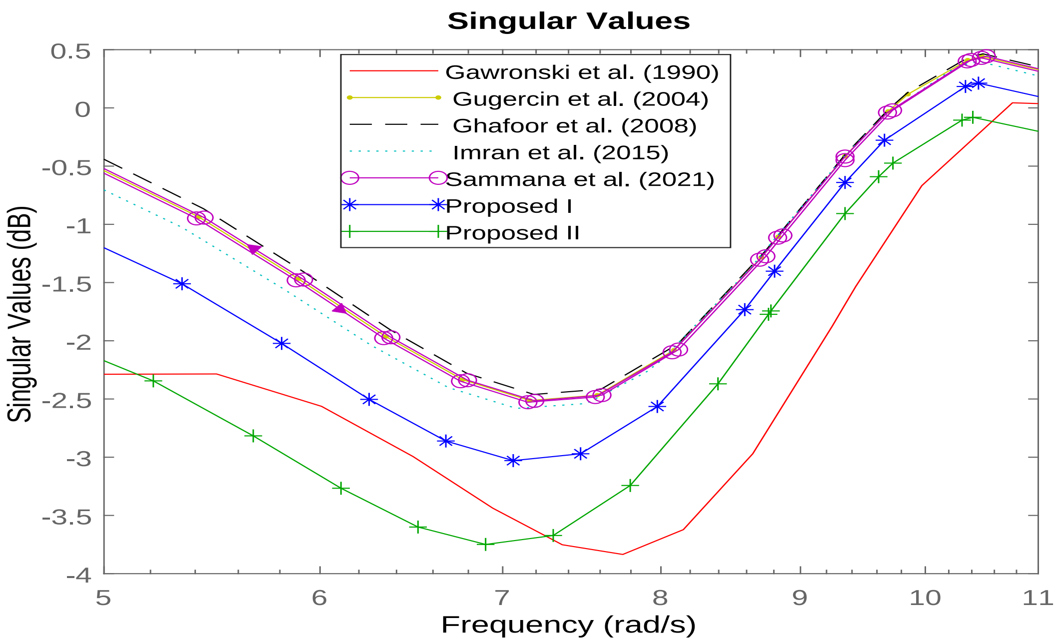

Figure 2. Close-up view of $\sigma[G_8(s) - G_1(s)]$ in $[\omega_1, \omega_2] = [5, 11]$

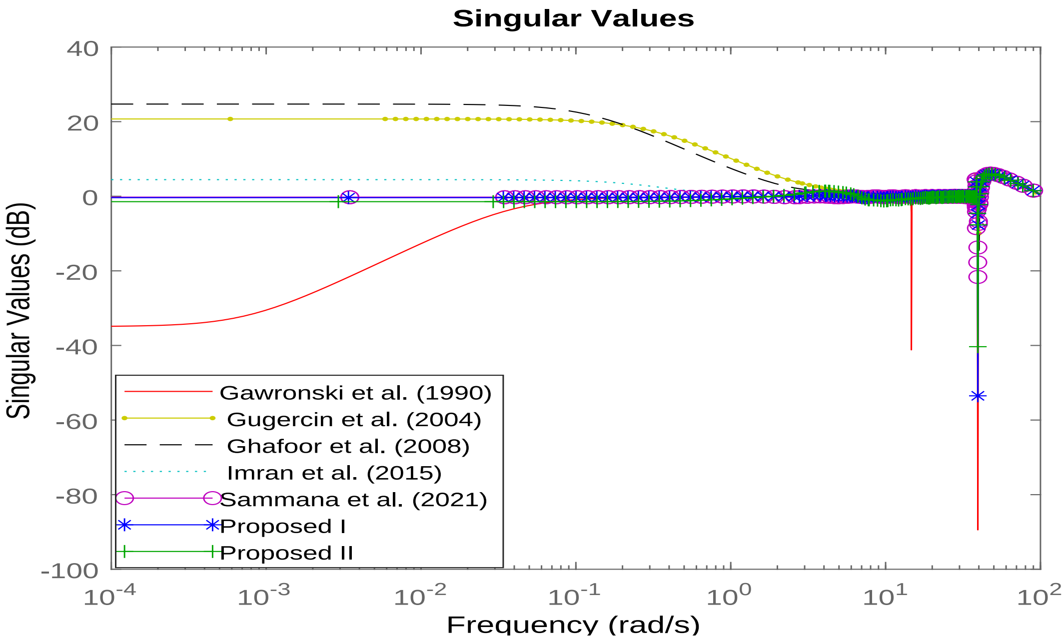

Figure 3. Frequency reponse error comparison $\sigma[G_{50}(s) - G_3(s)]$

sired frequency range $[\omega_1,\ \omega_2] = [1,\ 9] rad/s$, a close-up view of the error plot is illustrated in Fig. 6 to show the efficacy of the results of the proposed technique. It can be noted that the proposed techniques capitulate comparable approximation error as compared to *Gawronski et al.* (Gawronski & Juang, 1990) technique and compute low approximation error as compared to stability preservation techniques (Batool et al., 2021; Ghafoor & Imran, 2015; Ghafoor & Sreeram, 2008) within the desired frequency

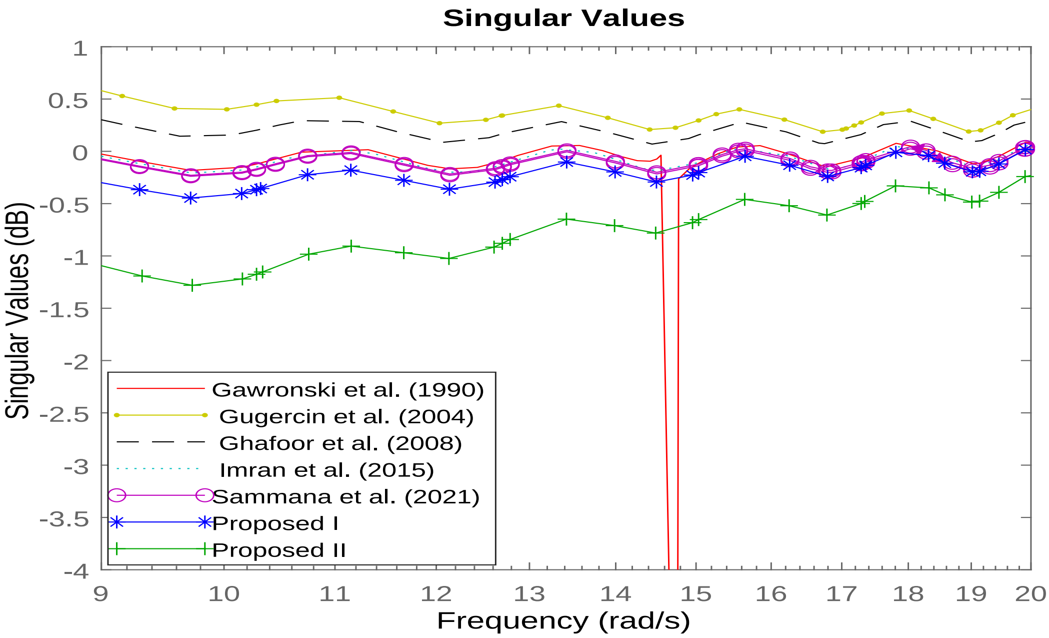

Figure 4. Close-up view of $\sigma[G_{50}(s) - G_3(s)]$ in $[\omega_1, \omega_2] = [9, 20]$

range.

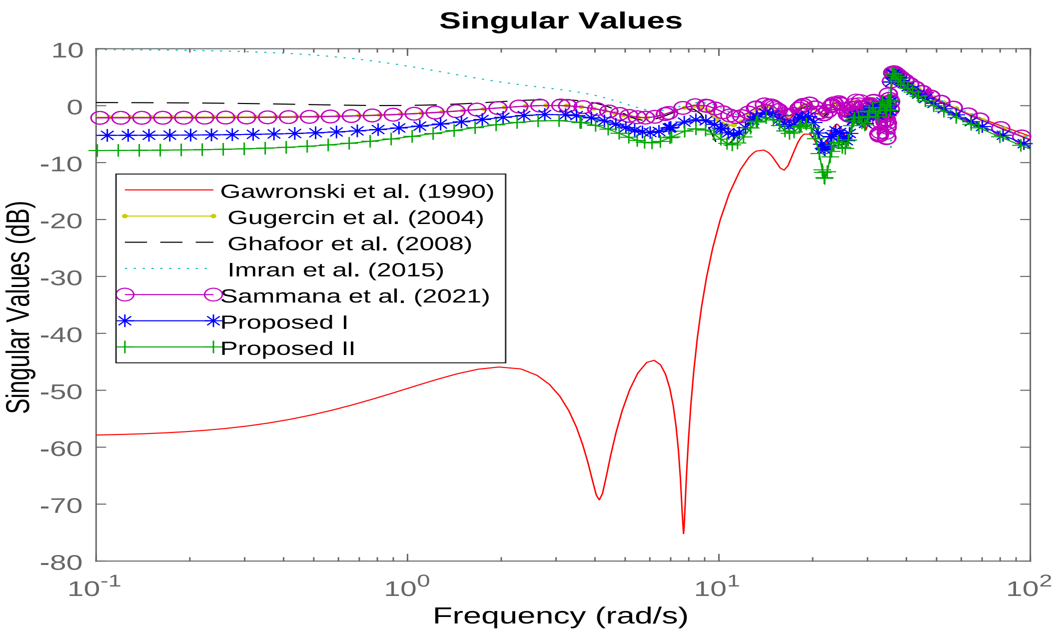

Figure 5. Frequency reponse error comparison $\sigma[G_{20}(s) - G_5(s)]$

Analysis & Discussion:

The efficacy of the proposed improved model-reduction techniques is evident in Tables 1 and 2, where the resulting ROMs exhibit notably lower approximation errors in comparison to established stability-conserving model-reduction techniques designed

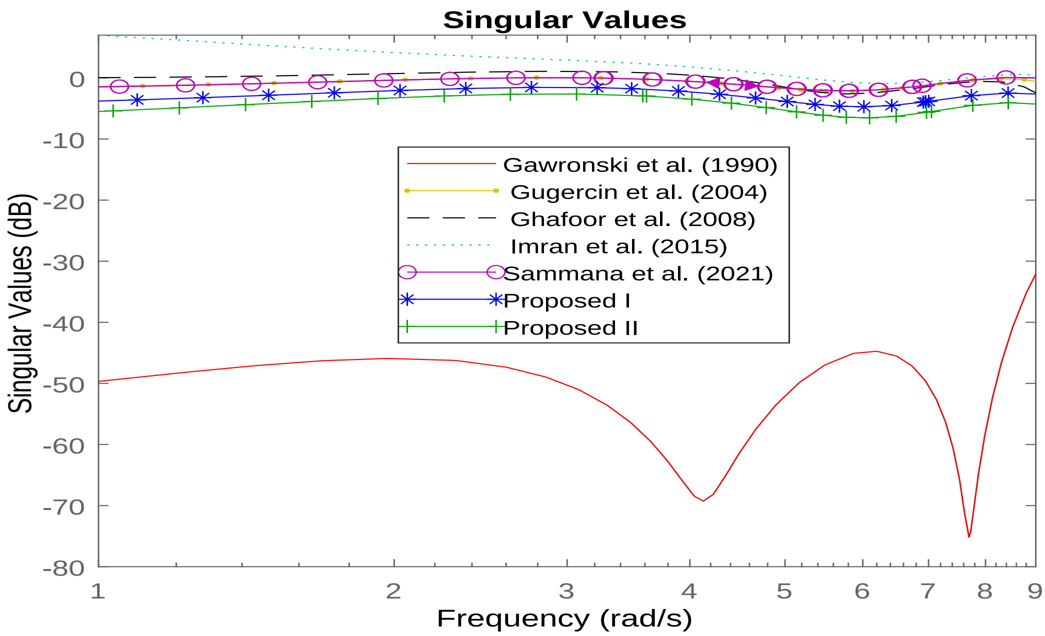

Figure 6. Close-up view of $\sigma[G_{20}(s) - G_5(s)]$ in $[\omega_1, \omega_2] = [1, 9]$

for frequency-weighted scenarios (Batool et al., 2022; Imran et al., 2014; Wang et al., 1999). Additionally, the proposed techniques provide computationally straightforward a priori error bounds.

A comparative analysis between the results of the technique proposed by Enns (Enns, 1984) and our improved techniques reveals that while the Enns technique yields the least approximation error, it comes at the expense of producing unstable ROMs—an undesirable outcome. This emphasizes the significance of the proposed techniques in achieving a balance between low approximation error and stability. The visual representation of the results in Fig. 1, Fig. 2, Fig. 3, Fig. 4, Fig. 5, and Fig. 6 further supports the superiority of the proposed techniques. These figures illus-trate that proposed techniques consistently yield lower approximation errors compared to existing stability-conserving model-reduction techniques designed for frequency-limited-interval scenarios (Batool et al., 2021; Ghafoor & Imran, 2015; Ghafoor & Sreeram, 2008; Gugercin & Antoulas, 2004).
In summary, the results presented in this study underscore the effectiveness of the proposed improved techniques in achieving a favorable trade-off between low approximation error and stability, positioning them as promising advancements in the field of MOR for both frequency-weighted and frequency-limited-interval scenarios.

5. Conclusions

In conclusion, this paper introduces state-of-the-art advancements in model order reduction, specifically addressing scenarios involving frequency-weighted and frequency-limited interval Gramians in the continuous-time domain. The novel techniques put forth herein not only establish stable Reduced Order Models but also exhibit significantly lower approximation error when compared to prevalent stability-conserving methodologies. Moreover, the inclusion of easily computable a priori error bounds further enhances the practical utility of the proposed techniques. The simulation results presented in this study showcase the effectiveness of the proposed approaches, demonstrating their utility and competitiveness in comparison to existing model-reduction techniques. As a promising avenue for future research, the application of these improved techniques to discrete-time systems and bilinear systems holds potential for expanding the scope of their applicability and advancing the state of the art in model order reduction methodologies.

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Acronyms/Notation

In this article, the following Acronyms/Notation are used:

MOR	Model order reduction
ROM	Reduced order model
BT	Balanced Truncation
LTI	Linear Time In-Variant
HSV	Hankel singular-values
ROs	Reduced-Orders

6. Data Availability Statement

"Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.'

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