SHANAEV, S. and GHIMIRE, B. 2021. *A fitting return to fitting returns: cryptocurrency distributions revisited*. [Working paper]. SSRN [online]. Available from: https://doi.org/10.2139/ssrn.3847351

A fitting return to fitting returns: cryptocurrency distributions revisited.

SHANAEV, S. and GHIMIRE, B.

2021

A fitting return to fitting returns:

Cryptocurrency distributions revisited

Savva Shanaev* and Binam Ghimire

University of Northumbria at Newcastle, United Kingdom

*corresponding author: s.shanaev@northumbria.ac.uk

Abstract

This study fits 22 theoretical distribution functions, four of them originally derived, onto 772 cryptocurrency daily returns with goodness-of-fit evaluated using Cramer-von Mises, Anderson-Darling, Kuiper, Kolmogorov-Smirnov, and Chi-squared tests, as well as a harmonic mean p-value synthetic criterion. Most cryptocurrency return distributions can be sufficiently approximated with a Johnson SU function or an asymmetric power function. Johnson SU, asymmetric Student, and asymmetric Laplace distributions have better fit for larger cryptocurrencies, while error, generalised Cauchy, and Hampel (a Gaussian-Cauchy mixture) distributions are more characteristic of smaller cryptocurrencies, with larger coins demonstrating better overall fit. Less than 8% of sample coins and less than 4% of the top quartile by size do not fit into any of the investigated distributions, three largest “misbehaving” cryptocurrencies being Litecoin, Dogecoin, and Decred. Bitcoin and Ethereum are best modelled with error and asymmetric power law distributions, respectively, with asymmetric power law distributions stable through time. More than 30% of sample cryptocurrencies, and 26% from the top quartile, have infinite theoretical variance, severely limiting the diversification potential with such cryptoassets. Three most prominent infinite-variance coins are Bitcoin SV, Tezos, and ZCash. This study has substantial implications for risk management, portfolio management, and cryptocurrency derivative pricing.

Keywords: cryptocurrency, return distribution, skewness, kurtosis, goodness-of-fit

JEL codes: C4, C22, C46, C49

1. Introduction and Literature Review

The riskiness of cryptocurrency investment has been long acknowledged in empirical finance research and has been often associated with thick-tailed and heavy-peaked distributions of cryptoasset returns (Osterrieder, 2017; Fry, 2018; Zhang et al., 2018; Szczygielski et al., 2020). Nevertheless, formal and rigorous research on theoretical distribution function fitting to empirical distributions of coins has remained scarce to date. This is especially important given voluminous research on diversification properties of cryptoassets (Briere et al., 2015; Bouri et al., 2017; Guesmi et al., 2019) and the recent advent of cryptocurrency derivatives, including futures (Corbet et al., 2018; Kochling et al., 2019), and options (Jalan et al., 2021).

Existing studies have mainly considered either very few candidate distributions (Osterrieder, 2017; Osterrieder and Lorenz, 2017; Punzo and Bagnato, 2021), focused solely on Bitcoin (da Cunha and da Silva, 2020), or simply documented the non-normality of cryptocurrency returns (Bariviera et al., 2017; Zhang et al., 2018; Takaishi, 2018).

The most rigorous piece of research on the topic to date is perhaps Szczygielski et al. (2020), who fit 58 candidate distribution to 15 most prominent cryptocurrencies and evaluate goodness-of-fit using Kolmogorov-Smirnov, Anderson-Darling, and Chi-squared tests. Their results reinforce the heavy tails of cryptocurrency return distributions, with Cauchy distribution fitting most cryptoassets the best, with error, Johnson SU, Burr, Dagum, and Laplace distributions also demonstrating best fit for individual coins. However, the null hypothesis of fit was often rejected even for such “winning” distributions. Most notably, the Cauchy distribution, declared the best fitting for Bitcoin, was rejected at 5% significance level for all three tests considered. Furthermore, Szczygielski et al. (2020) focus on largest coins only and do not show whether theoretical distributions remain stable through time. This study, therefore, seeks to address these notable gaps in the literature.

The rest of the paper is organised as follows. The next section briefly outlines the data, presents the stylised facts regarding cryptoasset empirical distribution function, and discusses the set of candidate distributions as well as the battery tests it applies to determine the goodness of fit. The findings chapter reports the estimation results, while the final section concludes.

2. Data and Methodology

2.1. The sample

This study has collected daily data on the exhaustive sample of cryptocurrencies from Coinmarketcap over the 2013-2019 period. For distribution fitting, daily logarithmic returns are calculated and days with no trading activity are excluded. Next, stablecoins and coins with less than 100 observations are removed from the sample. This resulted in a representative selection of 772 coins and 688,860 coin-day observations, the largest sample to date considered in cryptocurrency distribution research. Additionally, the study also retrieves full Bitcoin and Ethereum price history from July 2010 until December 2020 and from August 2015 until December 2020, respectively, to test for the stability of distributions for these two most prominent cryptoassets.

2.2. Stylised facts

Table 1 below reports the descriptive statistics for empirical cryptocurrency return distributions. To better understand the distribution shapes, skewness, excess kurtosis, and Tukey lambda is calculated for all sample coins and compared to those of most notable theoretical distribution functions. Tukey lambda (Tukey, 1962; Joiner and Rosenblatt, 1971) is a useful heuristic that can help the study identify the appropriate set of theoretical distributions to consider in goodness-of-fit estimations. Following Tukey (1962), the lambda values have been calculated on quantile-on-quantile plots by maximising the probability plot correlation coefficient between the empirical quantile function and the Tukey lambda quantile function:

$$Q_{Tukey} = \begin{cases} \frac{1}{\lambda}\left(p^{\lambda} - (1-p)^{\lambda}\right), \lambda \neq 0 \\ \ln \frac{p}{1-p}, \lambda = 0 \end{cases}$$

All but five (99.35%) of sample coins are right-skewed and all (100%) have positive excess kurtosis. However, the magnitudes of the moments estimated vary substantially, even among the largest cryptocurrencies. Only four return distributions are even remotely Gaussian, with excess kurtosis less than three and skewness between -0.5 and 0.5. Jarque and Bera (1980) test statistics $JB = \frac{nS^2}{6} + \frac{n(K-3)^2}{24} \sim \chi^2(2)$ for estimated values of skewness and kurtosis S and K showed significant deviations from normality at 5% for all sample coins, and for all but two coins (BitcoinHD and Creditcoin) at 1%, consistent with the established stylised facts reinforcing the non-normality of cryptoasset return distributions (Zhang et al., 2018).

Only three cryptocurrencies (Qwertycoin, eXPerience, and INDINODE) have a positive Tukey lambda, suggesting that distribution tails for 99.6% of sample coins are at least logistic or thicker. The Tuckey lambda value for Bitcoin is -0.26, very close to -0.2 reported by Da Cunha and da Silva (2020). The Cauchy distribution, commonly resorted to in the literature to model cryptocurrency returns (Fry, 2018; Szczygielski et al., 2020), is seemingly too heavy-tailed for cryptocurrency returns, with less than 8% of sample coins having a Tukey lambda of -1 or lower. 80% of cryptocurrencies fall within the interval from -0.88 to -0.23, representing the “grey area” between Laplace (-0.14) and Cauchy (-1.00) distributions. This potentially explains the puzzle Szczygielski et al. (2020) encountered and justifies considering a wide range of generalised distribution families, such as Student’s T, Johnson SU, and error, that could produce such Tukey lambdas for some shape parameter values. Additionally, prominent skewness of cryptocurrency returns warrants the use of asymmetric variations for all distributions considered if such exist.

Table 1. Descriptive statistics.

		Skewness	Excess kurtosis	Tukey lambda
individual coins	Bitcoin	0.50	9.65	-0.26
	Ethereum	0.29	14.41	-0.29
	XRP	6.30	107.95	-0.44
	Bitcoin Cash	1.69	10.17	-0.31
	Bitcoin SV	3.15	24.52	-0.54
	Litecoin	4.65	65.22	-0.41
sample characteristics	Mean	6.75	131.92	-0.58
	Minimum	-1.01	0.37	-8.26
	1st decile	1.08	5.95	-0.88
	2nd decile	1.72	9.53	-0.65
	3rd decile	2.26	15.02	-0.55
	4th decile	2.95	22.11	-0.47
	Median	3.94	34.23	-0.41
	6th decile	5.55	59.33	-0.37
	7th decile	7.47	97.36	-0.32
	8th decile	10.26	155.12	-0.27
	9th decile	16.24	358.41	-0.23
	Maximum	45.67	2116.23	0.12
notable theoretical distributions	Uniform	0.00	-1.20	-1.00
	Normal	0.00	0.00	0.14
	Logistic	0.00	1.20	0.00
	Hypersecant	0.00	2.00	-0.06
	Laplace	0.00	3.00	-0.12
	Cauchy	undefined	undefined	-1.00

2.3. Distributions

In this subsection, $F(x)$ and $f(x)$ represent the cumulative probability and probability density function of a theoretical distribution of interest; a, b, c, and d denote location, scale, shape, and asymmetry parameters, respectively; erf is the error function; Γ is the gamma function; γ is the lower incomplete gamma function; ${}_2F_1$ is the hypergeometric function; while Φ and φ are the cumulative and the probability density functions of the standard normal distribution.

All parameters are calibrated via a maximum likelihood estimation using the Nelder-Mead algorithm, unlike the existing studies that prioritised computational intensity of the methods and often resorted to the method of moments (Sczcygielski et al., 2020). Alternative calibrations with Powell and SLSQP optimisation procedures performed slightly worse in terms of goodness-of-fit and substantially worse in terms of computational efficiency. For distributions where method of moments parameter estimates are available, they were used as the starting values for the log-likelihood maximisation algorithm to improve convergence speed and precision. Raw data and code for the estimations are available upon request.

This study considered 22 theoretical distribution functions, most of them standard in empirical finance and risk management, while some, notably power, asymmetric power, generalised Cauchy, and Hampel distributions are originally derived and suggested to better represent the nature of cryptocurrency returns. Next, the distributions applied, their cumulative and probability density functions, and specific properties are discussed sequentially[1].

1) Normal distribution:

$$F(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x-a}{\sqrt{2}b}\right)$$

$$f(x) = \frac{1}{\sqrt{2\pi}b} e^{-\left(\frac{x-a}{\sqrt{2}b}\right)^2}$$

2) Asymmetric normal distribution:

$$F(x) = \begin{cases} \frac{d^2}{1+d^2}\left(1 + \operatorname{erf}\left(\frac{x-a}{\sqrt{2}bd}\right)\right), x \le a \\ 1 - \frac{1}{1+d^2}\left(1 - \operatorname{erf}\left(\frac{d(x-a)}{\sqrt{2}b}\right)\right), x > a \end{cases}$$

$$f(x) = \begin{cases} \frac{d}{\sqrt{2\pi}b(1+d^2)} e^{-\left(\frac{x-a}{\sqrt{2}bd}\right)^2}, x \le a \\ \frac{d}{\sqrt{2\pi}b(1+d^2)} e^{-\left(\frac{d(x-a)}{\sqrt{2}b}\right)^2}, x > a \end{cases}$$

3) Logistic distribution:

[1] Uniform, triangular, PERT, Levy, Frechet, Weibull, Rayleigh, raised cosine, gamma, and beta distributions were also considered, however they did not produce adequate fit for any of the sample cryptocurrencies and thus are omitted from the discussion for the sake of brevity.

$$F(x) = \frac{1}{1 + e^{-\frac{x-a}{b}}}$$

$$f(x) = \frac{e^{-\frac{x-a}{b}}}{b\left(1 + e^{-\frac{x-a}{b}}\right)^2}$$

4) Generalised logistic distribution:

$$F(x) = \left(1 + e^{-\frac{x-a}{b}}\right)^{-c}$$

$$f(x) = \frac{ce^{-\frac{x-a}{b}}}{b\left(1 + e^{-\frac{x-a}{b}}\right)^{c+1}}$$

5) Hyperbolic secant (hypersecant) distribution:

$$F(x) = \frac{2}{\pi}\operatorname{atan}\left(e^{\frac{\pi(x-a)}{2b}}\right)$$

$$f(x) = \frac{1}{2b}\operatorname{sech}\left(\frac{\pi(x-a)}{2b}\right)$$

6) Asymmetric secant distribution:

$$F(x) = \begin{cases} \frac{4d}{\pi(1+d^2)}\operatorname{atan}\left(e^{\frac{\pi(x-a)}{2bd}}\right), x \le a \\ 1 - \frac{4}{\pi(1+d^2)}\operatorname{atan}\left(e^{-\frac{\pi d(x-a)}{2b}}\right), x > a \end{cases}$$

$$f(x) = \begin{cases} \frac{d}{b(1+d^2)}\operatorname{sech}\left(\frac{\pi(x-a)}{2bd}\right), x \le a \\ \frac{d}{b(1+d^2)}\operatorname{sech}\left(\frac{\pi d(x-a)}{2b}\right), x > a \end{cases}$$

7) Laplace distribution (Laplace, 1986):

$$F(x) = \begin{cases} \frac{1}{2}e^{\frac{x-a}{b}}, x \le a \\ 1 - \frac{1}{2}e^{-\frac{x-a}{b}}, x > a \end{cases}$$

$$f(x) = \frac{1}{2b}e^{-\frac{|x-a|}{b}}$$

8) Asymmetric Laplace distribution:

$$F(x) = \begin{cases} \frac{d^2}{1+d^2} e^{\frac{x-a}{bd}}, x \leq a \\ 1 - \frac{1}{1+d^2} e^{-\frac{d(x-a)}{b}}, x > a \end{cases}$$

$$f(x) = \begin{cases} \frac{d}{b(1+d^2)} e^{\frac{x-a}{bd}}, x \leq a \\ \frac{c}{b(1+c^2)} e^{-\frac{d(x-a)}{b}}, x > a \end{cases}$$

Normal, logistic, hypersecant, and Laplace distributions, alongside their asymmetric variations, are among the most frequently used in empirical finance literature due to their simplicity. Normal and Laplace distributions have very compelling theoretical rationales to their finance applications, the first stemming from the central limit theorem, and the latter naturally emerging from a Gaussian process with exponentially distributed variance (Linden, 2001). Some attempts have been made to apply Laplace scale mixture to cryptocurrency distributions as well (Punzo and Bagnato, 2021), with Szczygielski et al. (2020) and Chan et al. (2017) showing Laplace achieves best fit in some tests for MIOTA, Monero, and MaidSafeCoin. However, this study suspects few cryptocurrencies would be best captured by these based on Tukey lambda values of $0.14, 0.00, -0.06$, and -0.14, respectively, as well as quite low theoretical excess kurtosis ($0, 1.2, 2$, and 3). Nevertheless, the asymmetric versions can still meaningfully contribute to the fit of the least heavy-tailed and moderately skewed cryptocurrencies.

9) Student (Student's T) distribution (Student, 1908):

$$F(x) = \frac{1}{2} + \frac{(x-a)\Gamma\left(\frac{c+1}{2}\right) {}_2F_1\left(\frac{1}{2}; \frac{c+1}{2}; \frac{3}{2}; -\frac{(x-a)^2}{cb^2}\right)}{b\sqrt{c\pi}\,\Gamma\left(\frac{c}{2}\right)}$$

$$f(x) = \frac{\Gamma\left(\frac{c+1}{2}\right)}{b\sqrt{c\pi}\,\Gamma\left(\frac{c}{2}\right)} \left(1 + \frac{(x-a)^2}{cb^2}\right)^{-\frac{c+1}{2}}$$

10) Asymmetric Student distribution:

$$F(x) = \begin{cases} \frac{d^2}{1+d^2}\left(1+\frac{2(x-a)\Gamma\left(\frac{c+1}{2}\right){}_2F_1\left(\frac{1}{2};\frac{c+1}{2};\frac{3}{2};-\frac{(x-a)^2}{b^2cd}\right)}{bd\sqrt{c\pi}\Gamma\left(\frac{c}{2}\right)}\right), x \le a \\ 1-\frac{1}{1+d^2}\left(1-\frac{2d(x-a)\Gamma\left(\frac{c+1}{2}\right){}_2F_1\left(\frac{1}{2};\frac{c+1}{2};\frac{3}{2};-\frac{d(x-a)^2}{b^2c}\right)}{b\sqrt{c\pi}\Gamma\left(\frac{c_p}{2}\right)}\right), x > a \end{cases}$$

$$f(x) = \begin{cases} \frac{d}{d^2+1}\frac{\Gamma\left(\frac{c+1}{2}\right)}{b\sqrt{c\pi}\Gamma\left(\frac{c}{2}\right)}\left(1+\frac{(x-a)^2}{b^2cd}\right)^{-\frac{c+1}{2}}, x \le a \\ \frac{d}{d^2+1}\frac{\Gamma\left(\frac{c+1}{2}\right)}{b\sqrt{c\pi}\Gamma\left(\frac{c}{2}\right)}\left(1+\frac{d(x-a)^2}{b^2c}\right)^{-\frac{c+1}{2}}, x > a \end{cases}$$

The conventional Student's T distribution as well as its asymmetric parametrisation is a very flexible family that can approximate a wide range of empirical return distributions, from nearly Gaussian (when the degrees of freedom parameter c dictating the shape of the curve is high), to heavy-tailed and even infinite-variance distributions (when c is less than two). As such, Tukey lambda values of a T distribution with $c = 30, 4, 3, 2,$ and 1 are $\lambda = 0.10, -0.15, -0.25, -0.45,$ and -0.99, respectively, which has the power to capture most sample cryptocurrencies. The theoretical case of the Student's distribution applicability for financial time series stems from it being a solution for a Gaussian distribution with inverse gamma distributed variance (Praetz, 1972), and for cryptocurrency returns, it has been recommended by Osterrieder (2017).

11) Error (generalised normal) distribution (Nadarajah, 2005):

$$F(x) = \frac{1}{2} + \frac{\text{sign}(x-a)\gamma\left(\frac{1}{c};\frac{|x-a|^c}{b}\right)}{2\Gamma\left(\frac{1}{c}\right)}$$

$$f(x) = \frac{c}{2b\Gamma\left(\frac{1}{c}\right)}e^{-\frac{|x-a|^c}{b}}$$

12) Johnson SU distribution (Johnson, 1949):

$$F(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(a_2 + b_2 \operatorname{asinh}\left(\frac{x - a_1}{b_1}\right)\right)$$

$$f(x) = \frac{b_2 e^{-\frac{1}{2}\left(a_2 + b_2 \operatorname{asinh}\left(\frac{x-a_1}{b_1}\right)\right)^2}}{b_1\sqrt{2\pi}\sqrt{1 + \left(\frac{x - a_1}{b_1}\right)^2}}$$

Error and Johnson SU distributions are both generalisations of the standard Gaussian that allow for a wide range of distribution shapes. While these do not generate pathological infinite-variance distributions, they are still shown to enjoy very good fit for some cryptocurrencies (Chan et al., 2017; Szczygielski et al., 2020). As such, for the sample of Szczygielski et al. (2020), NEO is best described by a Johnson SU, while Monero and MIOTA are best represented by the error distributions for most of the tests.

13) Cauchy distribution (Poisson, 1824):

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\operatorname{atan}\left(\frac{x - a}{b}\right)$$

$$f(x) = \frac{1}{b\pi\left(1 + \left(\frac{x - a}{b}\right)^2\right)}$$

The Cauchy distribution is among the most frequently used to model cryptocurrency returns. As such, it is the best fit for the majority of cryptocurrencies in the sample of Szczygielski et al. (2020). The theoretical intuition behind Cauchy's applicability to cryptocurrency returns is perhaps best given by Fry (2018), who suggests a model of cryptocurrency returns shaped by the ratio of trading volume and liquidity, both variables independent and normally distributed, which naturally gives the Cauchy distribution. Nevertheless, Szczygielski et al. (2020) concede that Cauchy distributions are still significantly different from the empirical coin distributions, most notably for Bitcoin, it thus being the best fit among the distributions considered, yet still not a good fit. This is reinforced by Tukey lambda values – very few sample coins have lambdas close to or lower than -1, corresponding to the theoretical Cauchy distribution having more pathological tails than the empirical distribution.

14) Generalised Cauchy distribution (original derivation):

$$F(x) = \begin{cases} 1 - \frac{2}{\pi}\text{atan}\left(\left(1 + \frac{|x-a|}{b}\right)^{c}\right), x \le a \\ \frac{2}{\pi}\text{atan}\left(\left(1 + \frac{|x-a|}{b}\right)^{c}\right), x > a \end{cases}$$

$$f(x) = \frac{2c\left(1 + \frac{|x-a|}{b}\right)^{c-1}}{b\pi\left(1 + \left(1 + \frac{|x-a|}{b}\right)^{2c}\right)}$$

To address the puzzle of simultaneously best and relatively poor fit of Cauchy theoretical distribution in the literature, this study proposes a power law modification for a Cauchy function by introducing the shape parameter c. $c = 1$ gives the regular Cauchy, while $c > 1$ and $c < 1$ generate thinner or heavier tails, respectively. While such a distribution is still infinite mean and undefined variance for any value of c, it can produce Tukey lambda values in the desired range to improve goodness-of-fit. For example, shape parameters $c = 0.5, 0.75, 1, 1.5,$ and 2 return lambdas $\lambda - 2.01, -1.34, -1, -0.64,$ and -0.41, respectively.

15) Burr distribution (Singh and Maddala, 2008):

$$F(x) = 1 - \left(1 + \left(\frac{e^x - a}{b}\right)^{c}\right)^{-d}$$

$$f(x) = \frac{e^x cd\left(\frac{e^x - a}{b}\right)^{c-1}}{b\left(1 + \left(\frac{e^x - a}{b}\right)^{c}\right)^{d+1}}$$

16) Dagum distribution (Dagum, 1975):

$$F(x) = \left(1 + \left(\frac{e^x - a}{b}\right)^{-c}\right)^{-d}$$

$$f(x) = \frac{cde^x}{e^x - a} \frac{\left(\frac{e^x - a}{b}\right)^{cd}}{\left(\left(\frac{e^x - a}{b}\right)^c + 1\right)^{d+1}}$$

The use of Burr and Dagum distributions for cryptocurrency return modelling follows Szczygielski et al. (2020), who report these as best fits for Chainlink and Cardano, respectively.

17) Slash distribution (Rogers and Tukey, 1972):

$$F(x) = \begin{cases} \frac{1}{2}, \ x = a \\ \Phi\left(\frac{x-a}{b}\right) - b\frac{\varphi(0) - \varphi\left(\frac{x-a}{b}\right)}{x-a}, \ x \neq a \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2b\sqrt{2\pi}}, \ x = a \\ b\frac{\varphi(0) - \varphi\left(\frac{x-a}{b}\right)}{(x-a)^2}, \ x \neq a \end{cases}$$

Rogers and Tukey (1972) define and derive the slash distribution as a ratio between independent normal and uniform variates. This allows to modify the model of Fry (2018) for the unform distribution of cryptocurrency liquidity as well as generate lighter tails as compared to Cauchy while still preserving the infinite mean and undefined variance property.

18) Power distribution (original derivation):

$$F(x) = \begin{cases} \frac{1}{2}\left(1 + \left|\frac{x-a}{b}\right|\right)^{1-c}, x \leq a \\ 1 - \frac{1}{2}\left(1 + \left|\frac{x-a}{b}\right|\right)^{1-c}, x > a \end{cases}$$

$$f(x) = \frac{c-1}{2b}\left(1 + \left|\frac{x-a}{b}\right|\right)^{-c}$$

This is an originally developed distribution for the study of cryptocurrency markets. It has a mean of $\mu = a$ (if $c \geq 2$), variance $\sigma^2 = \frac{2b^2}{(c-2)(c-3)}$ (if $c \geq 3$), skewness $S = 0$ (if $c \geq 4$), and excess kurtosis $K = 6\frac{(c-2)(c-3)}{(c-4)(c-5)} - 3$ (if $c \geq 5$). This parametrisation allows to easily account for Paretian tails in cryptocurrency returns as well as represent both finite and infinite variance distributions. Additionally, when c tends to infinity, this function converges to exactly Laplace with mean a and scale $s = b/c$, which can be intuitively grasped from $\lim_{c\to+\infty} K = \lim_{c\to+\infty} 6\frac{(c-2)(c-3)}{(c-4)(c-5)} - 3 = 3$ and strictly proven using $\lim_{c\to\infty} \frac{c-1}{2sc}\left(1 + \frac{|x-a|}{sc}\right)^{-c} = \frac{1}{2s} e^{-\frac{|x-a|}{s}}$.

19) Asymmetric power distribution (original derivation):

$$F(x) = \begin{cases} \frac{d^2}{1+d^2}\left(1 + \left|\frac{x-a}{bd}\right|\right)^{1-c}, x \le a \\ 1 - \frac{1}{1+d^2}\left(1 + \left|d\frac{x-a}{b}\right|\right)^{1-c}, x > a \end{cases}$$

$$f(x) = \begin{cases} \frac{(c-1)d}{b(1+d^2)}\left(1 + \left|\frac{x-a}{bd}\right|\right)^{-c}, x \le a \\ \frac{(c-1)d}{b(1+d^2)}\left(1 + \left|d\frac{x-a}{b}\right|\right)^{-c}, x > a \end{cases}$$

The asymmetric power distribution uses an additional parameter d for asymmetry modelling, with $d > 1, 0 < d < 1$, and $d = 1$ generating left-skewed, right-skewed, and symmetric distributions, respectively, analogous to the asymmetric Laplace distribution. This allows to model distributions coherent with the stylised facts documented by prior studies, as such, da Cunha and da Silva (2020) reporting asymmetric power law scaling in Bitcoin distribution tails, with $c = 3.53$ for the right tail and $c = 3.01$ for the left tail, consistent with asymmetric power law distributions with finite-variance and infinite or undefined higher moments.

20) Gumbel distribution (Gumbel, 1941):

$$F(x) = e^{-e^{-\frac{x-a}{b}}}$$

$$f(x) = \frac{1}{b} e^{-\frac{x-a}{b} - e^{-\frac{x-a}{b}}}$$

21) Generalised extreme value (GEV) distribution (Jenkinson, 1955):

$$F(x) = \begin{cases} e^{-\left(1+\frac{c(x-a)}{b}\right)^{-\frac{1}{c}}}, c \ne 0 \\ e^{-e^{-\frac{x-a}{b}}} \qquad, c = 0 \end{cases}$$

$$f(x)=\begin{cases}\frac{1}{b}\left(1+\frac{c(x-a)}{b}\right)^{-\frac{c+1}{c}} e^{-\left(1+\frac{c(x-a)}{b}\right)^{-\frac{1}{c}}} & ,c\neq 0\\ \frac{1}{b}e^{-\frac{x-a}{b}-e^{-\frac{x-a}{b}}} & ,c=0\end{cases}$$

The consideration of extreme value distribution family, including Gumbel, Weibull, Frechet, and generalised extreme value for cryptocurrency return modelling has been a staple in early academic research on the topic (Osterrieder and Lorenz, 2017). This definition of the GEV function using a shape parameter c allows to model a spectrum between Gumbel, Frechet, and Weibull distributions (Jenkinson, 1955). $c = 0$ gives an unbounded Gumbel distribution, while positive and negative values of c generate distributions from Frechet and Weibull families bounded from below and above, respectively. The use of GEV in this study is motivated by Silahli et al. (2021) who use modified two-sided Weibull distributions to model cryptocurrency portfolio returns, and Osterrieder and Lorenz (2017), who apply the GEV function to Bitcoin returns.

22) Hampel distribution (original derivation):

$$F(x)=\frac{1}{2}+c\ \mathrm{erf}\left(\frac{x-a_1}{\sqrt{2}b_1}\right)+\frac{(1-c)}{\pi}\mathrm{atan}\left(\frac{x-a_2}{b_2}\right)$$

$$f(x)=\frac{c}{\sqrt{2\pi}b_1}e^{-\left(\frac{x-a_1}{\sqrt{2}b_1}\right)^2}+\frac{1-c}{\pi b_2\left(1+\left(\frac{x-a_2}{b_2}\right)^2\right)}$$

This distribution is an original derivation based on the insights of Hampel and Zurich (1998). It is a Gaussian-Cauchy mixture with separate location and scale parameters for the normal and Cauchy components and a shape parameter $0 \leq c \leq 1$ for the weight of the normal distribution in the mix[2]. For $c > 0$, it has infinite mean and undefined variance. The Hampel distribution

[2] Hampel and Zurich (1998) proposed a thought experiment with $c = 1 - 10^{-10}$ to demonstrate how rare events can manifest themselves in large samples and how heavy-tailed distributions can seem very well-behaved for a small number of observations. To this study's best knowledge, no prior piece of empirical research applied this concept to empirical distribution fitting.

therefore can serve as a useful tool to model return processes that combine periods of relative stability with explosive behaviour – a prominent stylised fact in empirical finance studies of cryptocurrency markets (Fry and Cheah, 2016; Fry, 2018; Zhang et al, 2018). Mixture distributions have been shown to fit financial time series remarkably well at least since Kon (1984), who implemented Gaussian-Gaussian mixtures to model time-varying distribution parameters of stock returns.

2.4. Goodness-of-fit tests

Following Szczygielski et al. (2020) and to address the varying concepts of goodness-of-fit measurement for additional robustness, this study considers a selection of multiple tests. In addition to Kolmogorov-Smirnov, Anderson-Darling, and Chi-squared utilised by Szczygielski et al. (2020), this study also reports Kuiper test (as in Chan et al. 2017), and Cramer-von Mises test results. The formulae and testing procedures for all tests are outlined below:

1) Cramer-von Mises test (Cramer, 1928; Mises, 1939; Anderson, 1962):

$$CvM = \frac{1}{12n} + \sum_{i=1}^{n} \left(\frac{2i-1}{2n} - F(x_i) \right)^2 \sim N\left(0, \frac{1}{45}\right)$$

Cramer-von Mises test (Cramer, 1928; Mises, 1939) belongs to the goodness-of-fit test family also including the Watson test, and allows to intuitively test for the sum of squared deviations from the empirical distribution function.

2) Anderson-Darling test (Anderson and Darling, 1952):

$$AD = -n - \sum_{i=1}^{n} \frac{2i-1}{n} \left(\ln F(x_i) + \ln\left(1 - F(x_{n+1-i})\right) \right)$$

$$p_{AD} = \begin{cases} e^{1.29-5.71AD+0.0186AD^2}, & AD \geq 0.6 \\ e^{0.92-4.28AD-1.38AD^2}, & AD \in [0.34; 0.6) \\ 1 - e^{-8.32+42.80AD-59.94AD^2}, & AD \in [0.2; 0.34) \\ 1 - e^{-13.44+101.14AD-223.73AD^2}, & AD < 0.2 \end{cases}$$

Anderson-Darling test (Anderson and Darling, 1952) is a useful goodness-of-fit measure for risk-management purposes as it is shown to be most sensitive to distribution violations at the tails (Goldman and Kaplan, 2018). Therefore, the results obtained from the Anderson-Darling procedure can be especially relevant for risk management of cryptocurrency portfolios and value-at-risk calculations. The p-values for the test are calculated based on the exponential function approximation.

3) Kuiper test (Kuiper, 1960):

$$KP = \sqrt{n}\left(\max_i\left(\frac{i}{n} - F(x_i)\right) + \max_i\left(F(x_i) - \frac{i-1}{n}\right)\right)$$

$$p_{KP} = \sum_{t=1}^{+\infty} 2(4t^2KP^2 - 1)e^{-2t^2KP^2}$$

Kuiper test (1960) is a generalisation of the Kolmogorov-Smirnov statistic that treats maximum negative and positive deviations from the empirical distribution separately, having been applied to cryptocurrency distribution fitting by Chan et al. (2017). P-values for the test are calculated using the sum of the first ten terms of the infinite series.

4) Kolmogorov-Smirnov test (Kolmogorov, 1938; Smirnov, 1948; Massey, 1951):

$$KS = \sqrt{n}\max_i\left|\frac{i}{n} - F(x_i)\right|$$

$$p_{KS} = e^{-KS^2}$$

Kolmogorov-Smirnov (Kolmogorov, 1938; Smirnov, 1948; Massey, 1951) test assesses the goodness-of-fit using the supremum statistic based on the maximum absolute deviation from the empirical distribution function in either direction. Kolmogorov-Smirnov test is known to detect distribution violations more often at the hump of the distribution (Goldman and Kaplan, 2018).

5) Chi-squared test:

$$m = \lceil 1 + \log_2 n \rceil$$

$$\chi^2 = \sum_{i=1}^{m} \frac{(O_i - E_i)^2}{E_i} \sim \chi^2(m-1)$$

Following Szczygielski et al. (2020), this study defines the number of bins m for Chi-squared tests as the ceiling of one plus the base two logarithm of the number of observations to maximise the power of the test.

To assess the goodness-of-fit synthetically across five tests, this study opts to use the Wilson (2019) harmonic mean p-value $p_{HM} = \frac{5}{\frac{1}{p_{CvM}}+\frac{1}{p_{AD}}+\frac{1}{p_{KP}}+\frac{1}{p_{KS}}+\frac{1}{p_{\chi^2}}}$ commonly used for family-wise error rate adjustments in multiple testing when hypotheses are not independent, which is obviously the case when comparing p-values from different tests for the same empirical distribution. From this point onwards, the study refers to harmonic mean p-value results when assessing the overall fit if not specified otherwise.

3. Findings and Discussion

3.1. Individual goodness-of-fit tests

This section presents estimation results and goodness-of-fit across all tests. Tables 2-6 below report the metrics across Cramer-von Mises, Anderson-Darling, Kuiper, Kolmogorov-Smirnov, and Chi-squared tests, respectively.

Table 2. Goodness-of-fit across distributions: Cramer-von Mises test.

Distribution	best fits					fits (at 5%)	
	Total	1st quartile	2nd quartile	3rd quartile	4th quartile	number	% of coins
normal	0	0	0	0	0	13	1.68
asymmetric normal	2	0	2	0	0	14	1.81
logistic	0	0	0	0	0	122	15.80
generalised logistic	0	0	0	0	0	127	16.45
hyperbolic secant	0	0	0	0	0	94	12.18
asymmetric secant	0	0	0	0	0	32	4.15
Laplace	2	1	0	0	1	369	47.80
asymmetric Laplace	12	6	4	1	1	389	50.39
Student	12	7	2	3	0	652	84.46
asymmetric Student	114	59	33	14	8	663	85.88
error	57	2	10	18	27	613	79.40
Johnson SU	130	49	40	23	18	710	91.97
Cauchy	2	0	0	0	2	494	63.99
generalised Cauchy	85	24	25	19	17	702	90.93
Burr	0	0	0	0	0	83	10.75
Dagum	0	0	0	0	0	7	0.91
slash	2	1	1	0	0	581	75.26
power	35	5	11	13	6	669	86.66
asymmetric power	128	23	27	45	33	728	94.30
Gumbel	0	0	0	0	0	0	0.00
GEV	0	0	0	0	0	7	0.91
Hampel	187	16	36	57	78	705	91.32
none	4	0	2	0	2	4	0.52

Table 3. Goodness-of-fit across distributions: Anderson-Darling test.

distribution	best fits					fits (at 5%)	
	total	1st quartile	2nd quartile	3rd quartile	4th quartile	number	% of coins
normal	0	0	0	0	0	1	0.13
asymmetric normal	0	0	0	0	0	1	0.13
logistic	0	0	0	0	0	16	2.07
generalised logistic	0	0	0	0	0	23	2.98
hyperbolic secant	0	0	0	0	0	17	2.20
asymmetric secant	0	0	0	0	0	0	0.00
Laplace	0	0	0	0	0	97	12.56
asymmetric Laplace	12	5	3	3	1	140	18.13
Student	5	2	0	2	1	317	41.06
asymmetric Student	130	64	33	21	12	435	56.35
error	45	2	7	14	22	203	26.30
Johnson SU	141	50	43	25	23	442	57.25
Cauchy	0	0	0	0	0	20	2.59
generalised Cauchy	59	13	22	10	14	389	50.39
Burr	3	1	1	0	1	16	2.07
Dagum	0	0	0	0	0	0	0.00
slash	0	0	0	0	0	27	3.50
power	21	5	6	6	4	277	35.88
asymmetric power	125	26	36	38	25	409	52.98
Gumbel	0	0	0	0	0	0	0.00
GEV	0	0	0	0	0	0	0.00
Hampel	83	12	14	27	30	322	41.71
none	148	13	28	47	60	148	19.17

Table 4. Goodness-of-fit across distributions: Kuiper test.

distribution	best fits					fits (at 5%)	
	total	1st quartile	2nd quartile	3rd quartile	4th quartile	number	% of coins
normal	0	0	0	0	0	9	1.17
asymmetric normal	0	0	0	0	0	8	1.04
logistic	0	0	0	0	0	69	8.94
generalised logistic	0	0	0	0	0	71	9.20
hyperbolic secant	0	0	0	0	0	63	8.16
asymmetric secant	0	0	0	0	0	19	2.46
Laplace	6	5	0	0	1	317	41.06
asymmetric Laplace	8	1	3	2	2	315	40.80
Student	38	20	10	6	2	577	74.74
asymmetric Student	63	36	13	9	5	575	74.48
error	95	9	17	31	38	586	75.91
Johnson SU	128	42	40	28	18	643	83.29
Cauchy	3	0	1	0	2	252	32.64
generalised Cauchy	98	34	33	21	10	667	86.40
Burr	0	0	0	0	0	51	6.61
Dagum	0	0	0	0	0	3	0.39
slash	0	0	0	0	0	319	41.32
power	59	14	15	18	12	637	82.51
asymmetric power	65	6	9	23	27	633	81.99
Gumbel	0	0	0	0	0	0	0.00
GEV	0	0	0	0	0	4	0.52
Hampel	177	23	37	49	68	661	85.62
none	32	3	15	6	8	32	4.15

Table 5. Goodness-of-fit across distributions: Kolmogorov-Smirnov test.

distribution	best fits					fits (at 5%)	
	total	1st quartile	2nd quartile	3rd quartile	4th quartile	number	% of coins
normal	0	0	0	0	0	93	12.05
asymmetric normal	0	0	0	0	0	96	12.44
logistic	1	1	0	0	0	330	42.75
generalised logistic	1	0	1	0	0	336	43.52
hyperbolic secant	0	0	0	0	0	226	29.27
asymmetric secant	1	0	0	1	0	200	25.91
Laplace	5	2	1	2	0	543	70.34
asymmetric Laplace	13	5	4	3	1	547	70.85
Student	27	13	7	4	3	761	98.58
asymmetric Student	99	44	24	21	10	757	98.06
error	59	4	10	19	26	721	93.39
Johnson SU	152	54	47	32	19	764	98.96
Cauchy	8	0	1	2	5	694	89.90
generalised Cauchy	92	29	31	15	17	761	98.58
Burr	2	1	1	0	0	237	30.70
Dagum	1	0	1	0	0	51	6.61
slash	3	0	1	1	1	730	94.56
power	24	4	8	6	6	739	95.73
asymmetric power	108	13	25	36	34	758	98.19
Gumbel	1	0	0	1	0	20	2.59
GEV	1	0	1	0	0	52	6.74
Hampel	174	23	30	50	71	748	96.89
none	0	0	0	0	0	0	0.00

Table 6. Goodness-of-fit across distributions: Chi-squared test.

distribution	best fits					fits (at 5%)	
	Total	1st quartile	2nd quartile	3rd quartile	4th quartile	number	% of coins
normal	1	0	1	0	0	14	1.81
asymmetric normal	0	0	0	0	0	14	1.81
logistic	1	0	1	0	0	111	14.38
generalised logistic	0	0	0	0	0	107	13.86
hyperbolic secant	0	0	0	0	0	96	12.44
asymmetric secant	0	0	0	0	0	46	5.96
Laplace	3	0	2	1	0	344	44.56
asymmetric Laplace	12	4	3	4	1	357	46.24
Student	34	15	12	5	2	515	66.71
asymmetric Student	78	42	22	5	9	524	67.88
error	69	5	8	28	28	535	69.30
Johnson SU	66	27	19	13	7	586	75.91
Cauchy	33	4	6	11	12	384	49.74
generalised Cauchy	91	26	29	22	14	612	79.27
Burr	1	1	0	0	0	81	10.49
Dagum	0	0	0	0	0	6	0.78
slash	53	26	17	6	4	485	62.82
power	37	8	12	12	5	576	74.61
asymmetric power	96	11	25	31	29	602	77.98
Gumbel	0	0	0	0	0	3	0.39
GEV	0	0	0	0	0	6	0.78
Hampel	181	21	30	54	76	685	88.73
none	16	3	6	1	6	16	2.07

For all five tests, the selection of 22 functions employed by this study can describe the empirical cryptocurrency return distributions remarkably well. For Cramer-von Mises, Anderson-Darling, Kuiper, Kolmogorov-Smirnov, and Chi-squared tests, all null hypotheses were rejected only for 0.52% (1.30%), 19.71% (34.84%), 4.15% (5.44%), 0.00% (0.13%), and 2.07% (2.98%) of coins, respectively, on 5% (10%) significance level. The results strongly suggest that most distribution violations occur at the tails, which is crucial for risk management of cryptocurrency portfolios. The overwhelming majority of fits and best fits is achieved by a handful of distributions, mostly from the generalised families and the ones specifically derived in this study. As such, the asymmetric power, Johnson SU, Hampel, asymmetric Student, generalised Cauchy, and error distributions contribute to 99.35% (90.80%), 80.83% (75.52%), 95.85% (81.09%), 99.87% (88.60%), and 97.80% (75.26%) of fits (best fits) in the respective tests across the sample, implying that a relatively small set of theoretical functions is sufficient to model a wide range of empirical cryptocurrency distributions and highlighting the practical feasibility of implementing the approach of this study for investment practice.

Johnson SU, asymmetric Student, and generalised Cauchy distributions demonstrate better fit for larger coins, while asymmetric power, error, and Hampel distributions are more characteristic of smaller cryptocurrencies. Generalised asymmetric distributions overall perform much better than their symmetric counterparts, reinforcing that skewness is an important parameter to consider in cryptocurrency risk management. The extensions of the Cauchy distribution this study developed were successful in representing cryptocurrency returns, generalised Cauchy and Hampel mixture demonstrating very good fit, addressing the issues identified in the existing literature (Szczygielski et al., 2020). Burr, Dagum, and extreme value distributions, however, were not particularly well-performing, especially for Kuiper and Cramer-von Mises tests, contradicting the assertions of prior research. As expected from the analysis of stylised facts and descriptive stats, conventional distribution functions, such as normal, logistic, hypersecant, and Laplace, as well as their asymmetric versions, did not demonstrate good or best fits as well.

3.2. Harmonic mean p-value adjustment

Table 7 below reports the Wilson (2019) harmonic mean p-value to synthesise the results of five tests discussed above. The goodness-of-fit of asymmetric power, Johnson SU, asymmetric Student, Hampel, generalised Cauchy, and generalised normal (error) functions for empirical cryptocurrency return distributions is generally reinforced. For the harmonic mean p-value, the selection of these six distributions accounts for 92.36% (86.79%) of fits (best fits). Only 7.64% (12.69%) of coins overall, and only 3.63% (4.15%) from the top quartile, do not fit any of the 22 considered distributions at 5% (10%). The degree of overall fit is higher for more prominent coins, the largest "misbehaving" cryptocurrencies being Litecoin, Dogecoin, and Decred.

Table 7. Goodness-of-fit across distributions: Harmonic mean p-values.

distribution	best fits					fits (at 5%)	
	Total	1st quartile	2nd quartile	3rd quartile	4th quartile	number	% of coins
normal	0	0	0	0	0	3	0.39
asymmetric normal	0	0	0	0	0	5	0.65
logistic	0	0	0	0	0	41	5.31
generalised logistic	0	0	0	0	0	41	5.31
hyperbolic secant	0	0	0	0	0	40	5.18
asymmetric secant	0	0	0	0	0	0	0.00
Laplace	2	1	0	0	1	209	27.07
asymmetric Laplace	9	3	3	2	1	238	30.83
Student	4	1	0	3	0	465	60.23
asymmetric Student	131	67	34	20	10	522	67.62
Error	63	2	8	19	34	410	53.11
Johnson SU	143	51	42	26	24	565	73.19
Cauchy	0	0	0	0	0	77	9.97
generalised Cauchy	71	19	24	16	12	557	72.15
Burr	0	0	0	0	0	31	4.02
Dagum	0	0	0	0	0	1	0.13
Slash	1	1	0	0	0	107	13.86
power	27	5	9	7	6	476	61.66
asymmetric power	151	25	39	50	37	583	75.52
Gumbel	0	0	0	0	0	0	0.00
GEV	0	0	0	0	0	2	0.26
Hampel	111	11	18	35	47	513	66.45
none	59	7	16	15	21	59	7.64

Tables 8 and 9 below demonstrate the agreement between goodness-of-fit tests employed and the synthetic harmonic mean criterion in terms of both fits and best fits. In terms of fits, the tests are considered in agreement if they both accept or reject the null hypothesis of fit at the 5% confidence interval and in disagreement otherwise. Regarding best fits, the tests are considered in agreement if both suggest the same distribution (or none) to fit a particular cryptocurrency the best (with the highest p-value), and in disagreement otherwise. The agreement of a test with the harmonic mean criterion can be interpreted as the contribution of this test to the synthetic goodness-of-fit measure. This analysis can reveal which tests are the most informative when distinguishing between good and bad fits of theoretical functions to empirical cryptocurrency return distributions.

For fits, the highest agreement (>90%) is observed for Kuiper and Cramer-von Mises, Kuiper and Chi-squared, and Cramer-von Mises and Chi-squared, suggesting that using just one of the three aforementioned tests can suffice reasonably well when seeking to assess the fit of a single distribution to a cryptocurrency return time series. Kolmogorov-Smirnov and Anderson-Darling demonstrate the lowest agreement (58.95%), highlighting the notable stylised fact that they are more likely to reject the null at the hump and at the tails of the distribution, respectively (Goldman and Kaplan, 2018). The highest agreement with the harmonic mean criterion (~90%) is detected for Kuiper and Anderson-Darling, showing that a combination of these two tests can provide sufficient detail in terms of distribution fits. For best fits, in turn, a combination of Cramer-von Mises and Anderson-Darling could achieve a better result. The contribution of Kolmorogov-Smirnov is relatively low in both cases, implying relatively low discriminatory power of this simple test.

Table 8. Agreement between goodness-of-fit tests (fits).

Goodness-of-fit test	Cramer-von Mises	Anderson-Darling	Kuiper	Kolmogorov-Smirnov	Chi-squared	Harmonic mean
Cramer-von Mises	100.00	72.79	91.23	86.12	90.61	83.03
Anderson-Darling	72.79	100.00	80.36	58.95	78.80	89.72
Kuiper	91.23	80.36	100.00	78.53	91.22	90.40
Kolmogorov-Smirnov	86.12	58.95	78.53	100.00	79.52	69.16
Chi-squared	90.61	78.80	91.22	79.52	100.00	88.10
Harmonic mean	83.03	89.72	90.40	69.16	88.10	100.00

Table 9. Agreement between goodness-of-fit tests (best fits).

Goodness-of-fit test	Cramer-von Mises	Anderson-Darling	Kuiper	Kolmogorov-Smirnov	Chi-squared	Harmonic mean
Cramer-von Mises	100.00	59.59	50.78	58.29	40.03	70.98
Anderson-Darling	59.59	100.00	42.62	43.52	30.18	79.92
Kuiper	50.78	42.62	100.00	55.44	38.47	52.72
Kolmogorov-Smirnov	58.29	43.52	55.44	100.00	33.68	53.63
Chi-squared	40.03	30.18	38.47	33.68	100.00	38.47
Harmonic mean	70.98	79.92	52.72	53.63	38.47	100.00

3.3. Goodness-of-fit across sample coins

Table 10 below presents the best-fitting distributions across all five considered tests and the harmonic mean p-value for top 50 cryptocurrencies from the sample. Bitcoin and Ethereum are best described by error and asymmetric power distribution, respectively, both showing finite variance and reinforcing diversification properties of the largest cryptocurrencies (Briere et al., 2015; Guesmi et al., 2019). However, there are notable exceptions to this heuristic: Bitcoin SV, Tezos, ZCash, Ontology, Lisk, Monacoin, THETA, Horizen, V.Systems, and Bytom are shown to have infinite-variance distributions, while diversification properties of Litecoin, Dogecoin, and Decred are uncertain due to overall poor fit. Notably, ZCash, Monacoin, and Horizen has been prominent targets of 51% attacks, which can hint towards the link between malevolent attacks and tail risk on cryptocurrency markets (Shanaev et al., 2019; Grobys, 2021). Overall, only 62.31% of sample coins, and 69.95% from the first quartile, demonstrate finite variance, meaning investors seeking portfolio diversification need to carefully select cryptoassets with desired risk properties (see Table 11 below).

Table 11. Variance properties of cryptocurrency distributions, % of total.

theoretical variance	sample				
	Total	1st quartile	2nd quartile	3rd quartile	4th quartile
finite	62.31	69.95	64.77	58.55	55.96
infinite	30.05	26.42	26.94	33.68	33.16
uncertain	7.64	3.63	8.29	7.77	10.88

Table 10. Best fits for 50 largest cryptocurrencies across all tests (2013-2019).

Ticker	Kurtosis	Tukey lambda	Diversification benefits	Cramer-von Mises		Anderson-Darling		Kuiper		Kolmogorov-Smirnov		Chi-squared		Harmonic mean	
				best fit	p value	best fit	p value	best fit	p value	best fit	p value	best fit	p value	best fit	p value
BTC	9.65	-0.26	yes	ERR	0.6188	ERR	0.0703	ERR	0.5541	ERR	0.6235	APWR	0.9821	ERR	0.2384
ETH	14.41	-0.29	yes	APWR	0.8387	APWR	0.2603	APWR	0.9590	APWR	0.8281	APWR	0.9235	APWR	0.5976
XRP	107.95	-0.44	yes	APWR	0.6899	APWR	0.1182	PWR	0.5624	APWR	0.6516	APWR	0.8817	APWR	0.3451
BCH	10.17	-0.31	yes	APWR	0.8754	APWR	0.2540	PWR	0.9779	PWR	0.8035	PWR	0.9927	APWR	0.5959
BSV	24.52	-0.54	no	HPL	0.9130	HPL	0.2246	HPL	0.9899	HPL	0.8725	HPL	0.9970	HPL	0.5742
LTC	65.22	-0.41	uncertain	APWR	0.2882	none	0.0154	JSU	0.0572	JSU	0.4675	HPL	0.0693	none	0.0193
EOS	98.54	-0.42	yes	APWR	0.7492	APWR	0.1853	PWR	0.7997	JSU	0.6103	SLASH	0.9763	APWR	0.4078
BNB	24.76	-0.38	yes	APWR	0.8678	APWR	0.2914	JSU	0.9676	JSU	0.8421	PWR	0.9246	APWR	0.6288
XTZ	13.16	-0.26	no	GCAU	0.6140	GCAU	0.0624	LAP	0.2724	JSU	0.5543	SLASH	0.9778	GCAU	0.1941
ADA	71.39	-0.46	yes	JSU	0.7890	JSU	0.1266	GCAU	0.8703	STU	0.7408	SLASH	0.9984	JSU	0.3890
XMR	12.65	-0.23	yes	APWR	0.5887	APWR	0.0838	GCAU	0.4167	ALAP	0.5790	APWR	0.4639	APWR	0.2216
XLM	38.12	-0.36	yes	ASTU	0.6810	APWR	0.0636	ASTU	0.8264	ASTU	0.7875	ASTU	0.6399	ASTU	0.2265
TRX	38.68	-0.43	yes	GCAU	0.8120	APWR	0.1651	JSU	0.9462	GCAU	0.7393	CAU	0.8820	APWR	0.4431
ETC	485.54	-0.65	yes	APWR	0.6116	APWR	0.0968	GCAU	0.5827	JSU	0.6342	GCAU	0.2936	APWR	0.2042
DASH	263.81	-0.48	yes	ASTU	0.7098	ASTU	0.1040	ASTU	0.6518	ASTU	0.7326	JSU	0.4635	ASTU	0.2831
NEO	32.21	-0.38	yes	JSU	0.7501	ASTU	0.0987	JSU	0.9088	JSU	0.7856	STU	0.4687	ASTU	0.2725
ATOM	4.14	-0.26	yes	ASTU	0.6569	ASTU	0.1251	HPL	0.6948	HPL	0.7211	JSU	0.3753	ASTU	0.3165
MIOTA	6.07	-0.23	yes	APWR	0.8357	APWR	0.2865	PWR	0.9366	APWR	0.7840	SLASH	0.9895	APWR	0.6043
ZEC	146.05	-0.51	no	ASTU	0.7572	GCAU	0.1555	GCAU	0.9894	ASTU	0.8219	GCAU	0.9846	GCAU	0.4514
XEM	74.40	-0.36	yes	ASTU	0.5915	ASTU	0.0524	GCAU	0.9963	ASTU	0.6834	SLASH	0.6441	ASTU	0.1682
ONT	10.24	-0.30	no	ALAP	0.8448	GCAU	0.3084	GCAU	0.9868	JSU	0.8539	STU	0.9961	GCAU	0.6292
VET	7.50	-0.27	yes	JSU	0.9174	JSU	0.5108	JSU	0.9988	JSU	0.8702	JSU	0.9987	JSU	0.8065
DOGE	175.25	-0.46	uncertain	ASTU	0.3549	none	0.0127	ASTU	0.1052	ASTU	0.4159	none	0.0044	none	0.0155
ALGO	6.89	-0.38	yes	JSU	0.8486	ASTU	0.3149	STU	0.9610	HPL	0.8490	ALAP	0.9951	ASTU	0.6428
QTUM	16.68	-0.34	yes	JSU	0.7816	JSU	0.1748	HPL	0.5378	STU	0.6736	SLASH	0.2790	JSU	0.2789
DCR	7.52	-0.21	uncertain	JSU	0.5201	none	0.0092	JSU	0.6895	JSU	0.5446	GCAU	0.9502	none	0.0391
LSK	49.31	-0.49	no	ASTU	0.7426	ASTU	0.1661	ASTU	0.4298	ASTU	0.6136	ASTU	0.9989	ASTU	0.4057
ICX	8.63	-0.26	yes	JSU	0.9100	JSU	0.4295	GCAU	0.9989	GCAU	0.8576	STU	0.9990	JSU	0.7464
RVN	13.11	-0.27	yes	JSU	0.8520	JSU	0.2866	JSU	0.8983	JSU	0.8116	JSU	0.9657	JSU	0.6216
BTG	39.14	-0.51	yes	ASTU	0.8845	ASTU	0.3609	ASTU	0.9914	ASTU	0.8672	ASTU	0.9603	ASTU	0.7038
WAVES	6.45	-0.23	yes	ASTU	0.7199	ASTU	0.1093	ASTU	0.8058	ASTU	0.7614	ASTU	0.8738	ASTU	0.3511
BCD	169.48	-0.67	yes	JSU	0.8138	JSU	0.2151	ASTU	0.8593	ASTU	0.7880	JSU	0.9063	JSU	0.5237
MONA	34.79	-0.40	no	ASTU	0.5836	none	0.0305	JSU	0.6857	ASTU	0.7127	ASTU	0.2533	ASTU	0.1206
THETA	11.13	-0.27	no	GCAU	0.8981	GCAU	0.3113	LAP	0.9910	LAP	0.8704	SLASH	0.9828	GCAU	0.6579
NANO	14.22	-0.27	yes	ASTU	0.6123	APWR	0.0533	GCAU	0.9237	JSU	0.7265	ASTU	0.6738	ASTU	0.1872
SC	11.35	-0.27	yes	JSU	0.7675	JSU	0.0926	JSU	0.8941	JSU	0.7965	JSU	0.7533	JSU	0.3163
ZEN	135.76	-0.47	no	ALAP	0.8009	GCAU	0.1755	GCAU	0.9809	GCAU	0.7879	GCAU	0.9786	GCAU	0.4874
VSYS	30.19	-0.57	no	ASTU	0.8761	ASTU	0.4238	HPL	0.9955	HPL	0.8587	APWR	0.9342	ASTU	0.7299
BCN	256.15	-0.55	yes	HPL	0.5739	none	0.0369	HPL	0.4454	HPL	0.6306	HPL	0.7420	ASTU	0.1279
BTM	23.19	-0.37	no	APWR	0.7942	APWR	0.2102	JSU	0.9751	GCAU	0.7667	SLASH	0.9884	GCAU	0.4446
DGB	89.63	-0.40	yes	ASTU	0.7196	ASTU	0.1350	ASTU	0.4696	ASTU	0.6619	ASTU	0.3049	ASTU	0.3182
KMD	620.26	-0.96	yes	ASTU	0.6333	ASTU	0.0693	ASTU	0.5790	ASTU	0.6761	ASTU	0.4426	ASTU	0.2328
HC	17.17	-0.37	yes	ASTU	0.7619	ASTU	0.1513	ASTU	0.4244	HPL	0.6321	STU	0.5478	ASTU	0.3383
STEEM	62.49	-0.38	yes	JSU	0.7247	JSU	0.0964	STU	0.7600	JSU	0.7258	STU	0.7567	JSU	0.3078
BTS	13.45	-0.28	yes	GCAU	0.6406	APWR	0.0504	GCAU	0.8548	GCAU	0.7027	GCAU	0.8474	APWR	0.1744
IOST	227.47	-0.56	yes	APWR	0.7776	APWR	0.2276	APWR	0.5976	HPL	0.6901	ALAP	0.3598	APWR	0.3051
ZIL	4.14	-0.15	yes	JSU	0.8808	JSU	0.3079	ASTU	0.9877	JSU	0.8571	ASTU	0.9535	JSU	0.6499
XVG	230.40	-0.44	yes	JSU	0.6131	APWR	0.0526	PWR	0.5033	PWR	0.5927	SLASH	0.3458	APWR	0.1500
AION	22.86	-0.41	yes	ASTU	0.6833	ASTU	0.1100	HPL	0.3300	ASTU	0.5240	STU	0.9994	ASTU	0.2873
AE	15.60	-0.39	yes	JSU	0.8527	JSU	0.3134	JSU	0.8785	JSU	0.7918	SLASH	0.9856	JSU	0.6344

3.4. Dynamic goodness-of-fit for Bitcoin and Ethereum

Tables 12 and 13 below report harmonic mean p-values for goodness-of-fit on the full daily price history for Bitcoin and Ethereum, respectively, with estimations on the whole sample and in subsamples. Despite the error distribution generating the best fit for Bitcoin in 2013-2019, for the 2010-2020 sample period the asymmetric power distribution demonstrates significantly better fit, similarly to Ethereum. This highlights the flexibility of the asymmetric power function developed in this study and its robustness in subsamples. The asymmetric power distribution is also the only function showing high fit (all p-values greater than 20%) for all individual year periods and the whole sample for both coins, reinforcing the Paretian tails of largest cryptocurrency return distributions (da Cunha and da Silva, 2020).

Table 12. Stability of goodness-of-fit for Bitcoin (2011-2020).

distribution	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	Sample
normal	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
asymmetric normal	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
logistic	0.0010	0.0000	0.0000	0.0000	0.0003	0.0001	0.0000	0.0051	0.0000	0.0004	0.0001	0.0000
generalised logistic	0.0010	0.0000	0.0000	0.0000	0.0003	0.0001	0.0000	0.0111	0.0005	0.0004	0.0001	0.0000
hyperbolic secant	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0071	0.0002	0.0000	0.0000	0.0000
asymmetric secant	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Laplace	0.0169	0.0015	0.0011	0.0059	0.1522	0.0279	0.0007	0.2567	0.0076	0.1238	0.0423	0.0000
asymmetric Laplace	0.0945	0.0200	0.0014	0.0113	0.1457	0.0225	0.0007	0.3178	0.2297	0.1141	0.0711	0.0000
Student	0.1279	0.0040	0.1338	0.1126	0.3472	0.2347	0.2514	0.1121	0.0123	0.4298	0.2701	0.0000
asymmetric Student	0.3792	0.0303	0.3736	0.2118	0.3499	0.3054	0.3723	0.1556	0.0533	0.4563	0.4437	0.0000
error	0.0316	0.0052	0.0546	0.0566	0.5517	0.1274	0.1857	0.3136	0.0114	0.6516	0.1086	0.0002
Johnson SU	0.5065	0.1027	0.2864	0.3140	0.3555	0.2852	0.3954	0.2825	0.1096	0.4630	0.5304	0.0001
Cauchy	0.0864	0.0018	0.0244	0.0212	0.0290	0.0086	0.1112	0.0061	0.0017	0.0320	0.0126	0.0000
generalised Cauchy	0.0690	0.0064	0.1451	0.1216	0.6027	0.2204	0.3443	0.2083	0.0160	0.6775	0.2405	0.0000
Burr	0.0020	0.0000	0.0000	0.0000	0.0003	0.0002	0.0000	0.0052	0.0000	0.0006	0.0002	0.0000
Dagum	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
slash	0.1072	0.0055	0.0778	0.0566	0.0674	0.0404	0.2316	0.0123	0.0047	0.0802	0.0533	0.0000
power	0.0633	0.0046	0.1001	0.0596	0.7104	0.0843	0.3366	0.3035	0.0092	0.7384	0.1348	0.0002
asymmetric power	0.6041	0.3099	0.3235	0.2894	0.6946	0.2303	0.4197	0.3465	0.3142	0.6682	0.4277	0.2434
Gumbel	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GEV	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Hampel	0.2167	0.2694	0.1665	0.4349	0.3852	0.3495	0.1731	0.1547	0.4119	0.5342	0.2527	0.0000

Table 13. Stability of goodness-of-fit for Ethereum (2015-2020).

distribution	2015	2016	2017	2018	2019	2020	Sample
normal	0.0002	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000
asymmetric normal	0.0005	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000
logistic	0.0285	0.0000	0.0000	0.0272	0.0028	0.0319	0.0000
generalised logistic	0.0455	0.0000	0.0000	0.0425	0.0024	0.0235	0.0000
hyperbolic secant	0.0081	0.0000	0.0000	0.0742	0.0020	0.0002	0.0000
asymmetric secant	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Laplace	0.1238	0.0008	0.0012	0.5244	0.1571	0.3869	0.0002
asymmetric Laplace	0.6686	0.1100	0.0503	0.7383	0.1995	0.3002	0.0065
Student	0.1996	0.0021	0.0053	0.1616	0.2699	0.6223	0.0014
asymmetric Student	0.6992	0.1326	0.3139	0.1977	0.4215	0.6940	0.0427
error	0.1528	0.0005	0.0030	0.5184	0.2267	0.2879	0.0004
Johnson SU	0.7774	0.2330	0.4699	0.3038	0.3097	0.6744	0.0541
Cauchy	0.0292	0.0002	0.0010	0.0047	0.0059	0.0021	0.0000
generalised Cauchy	0.1666	0.0011	0.0045	0.3015	0.2955	0.5659	0.0027
Burr	0.1159	0.0013	0.0002	0.0278	0.0028	0.0287	0.0000
Dagum	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
slash	0.0810	0.0006	0.0034	0.0145	0.0257	0.0093	0.0000
power	0.0791	0.0006	0.0041	0.5178	0.2493	0.2701	0.0011
asymmetric power	0.8289	0.4323	0.3817	0.7413	0.4539	0.3017	0.4811
Gumbel	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GEV	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Hampel	0.6074	0.4117	0.0945	0.3097	0.1522	0.6490	0.0131

Table 14. Asymmetric power distribution parameters for Bitcoin and Ethereum.

Year	Bitcoin				Ethereum			
	location	scale	shape	asymmetry	location	scale	shape	asymmetry
2010	0.0014	0.1516	4.4715	1.2006				
2011	-0.0009	0.2123	5.5831	1.1799				
2012	0.0020	0.0489	4.1098	1.0814				
2013	0.0089	0.1561	5.4759	1.0870				
2014	-0.0017	0.1322	7.2039	0.9792				
2015	0.0007	0.0970	6.2491	1.0618	-0.0111	0.6796	13.6157	1.2149
2016	0.0018	0.0298	3.9848	1.0428	-0.0035	0.3167	9.1156	1.2372
2017	0.0085	0.7522	24.0365	0.9453	0.0030	0.2605	7.7210	1.2043
2018	0.0009	0.6045	22.4088	0.8677	-0.0060	0.1161	4.5432	0.8720
2019	0.0013	0.1105	6.7938	1.0394	-0.0008	0.3109	13.0495	1.0441
2020	0.0021	0.1039	6.6064	1.0848	0.0052	0.3185	11.8309	1.0125
Sample	0.0020	0.0713	4.2370	1.0799	-0.0002	0.2806	8.9746	1.0924

Table 14 above reports the estimated distribution parameters for the sample and individual year subsamples, showing time-varying asymmetry and shape of the distribution. Nevertheless, in all estimations both Bitcoin and Ethereum distribution remain finite-variance (shape parameter $c > 3$), further reinforcing their quality as diversifiers (Briere et al., 2015; Guesmi et al., 2019).

4. Conclusion

This study has applied 22 theoretical distributions functions, including four – power, asymmetric power, generalised Cauchy, and Hampel – originally derived, to empirical return distributions for a representative sample of 772 cryptocurrency markets, using a battery of five tests and a synthetic harmonic mean p-value criterion to assess goodness-of-fit.

The null hypothesis of fit has been accepted for 92.36% of sample cryptocurrencies, and 96.37% of coins from the top quartile by size. Most null hypothesis violations occur at the tails of the empirical distributions. The best fitting functions come from generalised families capable of producing heavy-tailed or skewed distributions, such as the asymmetric power, Johnson SU, Hampel, asymmetric Student, generalised Cauchy, and Hampel. This reinforces the stylised facts of heavy tails, non-normal distributions being appropriate for cryptocurrency modelling established by the existing literature. Nevertheless, the distribution selection utilised by this study achieves a sample-wide degree of goodness-of-fit unmatched in prior studies, which is the main contribution of this paper.

Johnson SU, asymmetric Student, and generalised Cauchy produce better overall fit for larger coins, whereas asymmetric power, Hampel, and error distributions perform comparatively better for smaller cryptocurrencies. These six distributions alone account for 92.36% (86.79%) of fits (best fits), while most information regarding fits (best fits) can be extracted from only two tests: Anderson-Darling and Kuiper (Cramer-von Mises), which allows to model cryptocurrency distributions feasibly and efficiently in the finance practice, be it in risk management or derivative pricing contexts. The largest coins with no identifiable theoretical distribution are Litecoin, Dogecoin, and Decred.

Only 62% of coins, and only 70% of the top quartile by size, reliably demonstrate finite theoretical variance. This has important implications for cryptocurrency investors and portfolio managers seeking to use cryptoassets as diversifiers. The most prominent infinite-variance coins in the sample are Bitcoin SV, Tezos, and ZCash. Bitcoin and Ethereum are consistently finite-variance, relatively stable with time, and can be best described by an asymmetric power function, reinforcing their value as suitable portfolio diversifiers as well as the phenomenon of Paretian tails on financial markets.

Further research on the topic could examine the distributions of cryptocurrency portfolios and test their stability, as well as apply the insights generated in this study for asset-pricing, option valuation, and risk management for cryptocurrency investment.

References

Anderson, T. (1962). On the distribution of the two-sample Cramer-von Mises criterion. *The Annals of Mathematical Statistics*, *33*(3), 1148-1159.

Anderson, T., & Darling, D. (1952). Asymptotic theory of certain" goodness of fit" criteria based on stochastic processes. *The annals of Mathematical Statistics, 23*(2), 193-212.

Bariviera, A., Basgall, M., Hasperue, W., & Naiouf, M. (2017). Some stylized facts of the Bitcoin market. *Physica A: Statistical Mechanics and its Applications, 484*(1), 82-90.

Bouri, E., Molnar, P., Azzi, G., Roubaud, D., & Hagfors, L. (2017). On the hedge and safe haven properties of Bitcoin: Is it really more than a diversifier? *Finance Research Letters, 20*(1), 192-198.

Briere, M., Oosterlinck, K., & Szafarz, A. (2015). Virtual currency, tangible return: Portfolio diversification with bitcoin. *Journal of Asset Management, 16*(6), 365-373.

Chan, S., Chu, J., Nadarajah, S., & Osterrieder, J. (2017). A statistical analysis of cryptocurrencies. *Journal of Risk and Financial Management*, *10*(2), 12.

Corbet, S., Lucey, B., Peat, M., & Vigne, S. (2018). Bitcoin futures—What use are they? *Economics Letters*, *172*(1), 23-27.

Cramer, H. (1928). On the composition of elementary errors: First paper: Mathematical deductions. *Scandinavian Actuarial Journal*, *1*(1), 13-74.

Da Cunha, C., & da Silva, R. (2020). Relevant stylized facts about bitcoin: Fluctuations, first return probability, and natural phenomena. *Physica A: Statistical Mechanics and its Applications*, *550*(1), 124155.

Dagum, C. (1975). A model of income distribution and the conditions of existence of moments of finite order. *Bulletin of the International Statistical Institute*, *46*(1), 199-205.

Fry, J. (2018). Booms, busts and heavy-tails: The story of Bitcoin and cryptocurrency markets? *Economics Letters*, *171*(1), 225-229.

Fry, J., & Cheah, E. (2016). Negative bubbles and shocks in cryptocurrency markets. *International Review of Financial Analysis*, *47*(1), 343-352.

Goldman, M., & Kaplan, D. (2018). Comparing distributions by multiple testing across quantiles or CDF values. *Journal of Econometrics*, *206*(1), 143-166.

Grobys, K. (2021). When the blockchain does not block: on hackings and uncertainty in the cryptocurrency market. *Quantitative Finance*.

Guesmi, K., Saadi, S., Abid, I., & Ftiti, Z. (2019). Portfolio diversification with virtual currency: Evidence from bitcoin. *International Review of Financial Analysis*, *63*(1), 431-437.

Gumbel, E. (1941). The return period of flood flows. *The Annals of Mathematical Statistics*, *12*(2), 163-190.

Hampel, F., & Zurich, E. (1998). Is statistics too difficult? *Canadian Journal of Statistics*, *26*(3), 497-513.

Jalan, A., Matkovskyy, R., & Aziz, S. (2021). The Bitcoin options market: A first look at pricing and risk. *Applied Economics*, *17*(1), 1-16.

Jarque, C., & Bera, A. (1980). Efficient tests for normality, homoscedasticity and serial independence of regression residuals. *Economics Letters*, *6*(3), 255-259.

Jenkinson, A. (1955). The frequency distribution of the annual maximum (or minimum) values of meteorological elements. *Quarterly Journal of the Royal Meteorological Society*, *81*(2), 158-171.

Johnson, N. (1949). Systems of frequency curves generated by methods of translation. *Biometrika*, *36*(1/2), 149-176.

Joiner, B., & Rosenblatt, J. (1971). Some properties of the range in samples from Tukey's symmetric lambda distributions. *Journal of the American Statistical Association*, *66*(2), 394-399.

Kochling, G., Muller, J., & Posch, P. (2019). Does the introduction of futures improve the efficiency of Bitcoin? *Finance Research Letters*, *30*(1), 367-370.

Kolmogorov, A. (1938). On the analytic methods of probability theory. *Proceeds of Mathematical Sciences*, 1(5), 5-41.

Kon, S. (1984). Models of stock returns—a comparison. *The Journal of Finance*, *39*(1), 147-165.

Kuiper, N. (1960). Tests concerning random points on a circle. In *Proceedings of the Netherlands Academy of Sciences* (Vol. 63, No. 1, pp. 38-47).

Laplace, P. (1986). Memoir on the probability of the causes of events. *Statistical science*, *1*(3), 364-378.

Linden, M. (2001). A model for stock return distribution. *International Journal of Finance & Economics*, *6*(2), 159-169.

Massey, F. (1951). The Kolmogorov-Smirnov test for goodness of fit. *Journal of the American statistical Association*, *46*(1), 68-78.

Mises, R. (1939). *Probability, statistics and truth.* Macmillan.

Nadarajah, S. (2005). A generalized normal distribution. *Journal of Applied Statistics, 32*(7), 685-694.

Osterrieder, J. (2017). The statistics of bitcoin and cryptocurrencies. In *2017 International Conference on Economics, Finance and Statistics* (pp. 285-289). Atlantis Press.

Osterrieder, J., & Lorenz, J. (2017). A statistical risk assessment of Bitcoin and its extreme tail behavior. *Annals of Financial Economics*, *12*(1), 1750003.

Poisson, S. (1824). Sur la probabilité des résultats moyens des observations. *Connaissance des tems*, 273-302.

Praetz, P. (1972). The distribution of share price changes. *Journal of Business*, *45*(1), 49-55.

Punzo, A., & Bagnato, L. (2021). Modeling the cryptocurrency return distribution via Laplace scale mixtures. *Physica A: Statistical Mechanics and its Applications*, *563*, 125354.

Rogers, W., & Tukey, J. (1972). Understanding some long-tailed symmetrical distributions. *Statistica Neerlandica*, *26*(3), 211-226.

Shanaev, S., Shuraeva, A., Vasenin, M., & Kuznetsov, M. (2019). Cryptocurrency value and 51% attacks: evidence from event studies. *The Journal of Alternative Investments*, *22*(3), 65-77.

Silahli, B., Dingec, K. D., Cifter, A., & Aydin, N. (2021). Portfolio value-at-risk with two-sided Weibull distribution: Evidence from cryptocurrency markets. *Finance Research Letters, 38*(1), 101425.

Singh, S., & Maddala, G. (2008). A function for size distribution of incomes. In *Modeling income distributions and Lorenz curves* (pp. 27-35). Springer, New York, NY.

Smirnov, N. (1948). Table for estimating the goodness of fit of empirical distributions. *The annals of mathematical statistics*, *19*(2), 279-281.

Student (1908). The probable error of a mean. *Biometrika*, *6*(1). 1-25.

Szczygielski, J., Karathanasopoulos, A., & Zaremba, A. (2020). One shape fits all? A comprehensive examination of cryptocurrency return distributions. *Applied Economics Letters*, *27*(19), 1567-1573.

Takaishi, T. (2018). Statistical properties and multifractality of Bitcoin. *Physica A: statistical mechanics and its applications*, *506*(1), 507-519.

Tukey, J. (1962). The future of data analysis. *Annals of Mathematical Statistics*, *33*(1), 1-67.

Wilson, D. (2019). The harmonic mean p-value for combining dependent tests. *Proceedings of the National Academy of Sciences*, *116*(4), 1195-1200.

Zhang, W., Wang, P., Li, X., & Shen, D. (2018). Some stylized facts of the cryptocurrency market. *Applied Economics*, *50*(55), 5950-5965.