An investigation into the use of conformal transformations in two-dimensional field theory.

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AN INVESTIGATION INTO THE USE OF CONFORMAL TRANSFORMATIONS IN TWO-DIMENSIONAL FIELD THEORY.

by

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ABSTRACT

The object of this thesis is to investigate the use of Conformal Transformations in solving Laplace's equation for two-dimensional field problems. Conformal Transformations is one of several analytical methods available for this purpose and its usefulness is generally limited to those problems in which the transformation integral is solvable.

The theory of Conformal Transformations and the various methods of solution are given along with two examples which illustrate the different boundary conditions. Various field problems that can be obtained from the transformation equation are also included. This provides the information necessary to tackle specific problems. The main section of the thesis involves the investigation of several corner configurations which illustrate the use of the different methods of solution and also the degree of manipulation often required in some of the problems. In some cases a field plot is made of the area around the corner while others contain a graph of the field strength variations along the conductor surface. In conclusion an outline of two non-analytical methods is given.

The study provides a step by step account of the procedure involved in solving, from first principles, the required field properties of particular problems, most of which have not previously been investigated.

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CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Field Theory

Field theory was developed in the 19th Century by such eminent mathematicians as Gauss, Laplace and Poisson. For over a century the theory has constituted an important branch of mathematical physics but only in recent decades has it been applied energetically to the solution of engineering and similar problems. There is a growing realization that field theory, as a tool in the hands of the engineer and scientist, can play a major part in the analysis of such phenomena as magnetostatics, heat conduction, fluid flow, electrostatics, elasticity and electromagnetic waves.

Many physical applications of field theory are unified by the fact that they can be expressed in partial differential equations containing the Laplacian. For example we have:

Laplace's equation $\nabla^2 V = 0$ -------- (1.1)

Poisson's equation $\nabla^2 V = -K$ -------- (1.2)

The diffusion equation $\nabla^2 V = (1/h^2)(\partial V/\partial t)$ ------------- (1.3)

The wave equation $\nabla^2 V = (1/c^2)(\partial^2 V/\partial t^2)$ ------------- (1.4)

In electrical problems V represents the electric potential; in the magnetic case, the magnetic potential; in the thermal case, the temperature. K represents a known function of position or a constant while the symbols h^2 and c^2 are parameters representing properties of the medium.

Laplace's equation applies to electrostatics, magnetostatics and the steady state flow of electricity, heat and incompressable fluids. Poisson's equation appears for example in determining the field inside a vacuum tube where a cloud of electrons produces a space charge. One of the uses of the diffusion equation is in calculating thermal transients while the wave equation applies to acoustic and electromagnetic waves.

The simplest of these equations is the Laplace equation and since this thesis deals mainly with electric and magnetic fields, it is the solution of this equation that interests us.

1.2 Methods of solving Laplace's Equation.

Laplace's equation can be solved in a number of ways. Often the most satisfactory solution is an exact mathematical one, but when an exact solution is not obtainable we may resort to a numerical approximation or to a graphical or an experimental solution. The most commonly used methods are listed here.

1. Analytical - This usually involves an exact mathematical solution and can be subdivided into various mathematical methods, examples of two of which are given here.

1(a) Separation of Variables: In this method the partial differential equations (1.1) to (1.4) are broken down into ordinary differential equations and the final solution is built up from particular solutions of these ordinary differential equations. This procedure can be used with any number of independent variables and is applicable to cylindrical and axially symmetrical fields.

1(b) Functions of a Complex Variable: The method of obtaining a solution of Laplace's equation using functions of a complex variable is sometimes referred to as the method of conjugate functions or the method of conformal transformations. It involves transforming a rectangular mesh in one complex plane into the desired field map in another complex plane. It is handicapped however in that it can be applied only to cylindrical fields of one or two independent variables.

2. Numerical Approximation - Employing the 'relaxation' method, numerical approximation consists in replacing the smooth potential variation by a set of discrete values at the intersection of a grid. It is an iterative method relying for accuracy on tedious numerical analysis, and is limited ordinarily to fields in which the potential is a function of only two space variables.

3. Graphical - This method, suitable for two-dimensional fields, requires no knowledge of mathematics and consists of a map of the region showing equipotentials and lines of flow, which divide the entire region into curvilinear squares. The procedure consists of sketching a few equipotentials and lines of flow on a scale drawing of the region, then subdividing the resulting curvilinear squares until the required accuracy is obtained. The method of

images, where imaginary points or line charges are introduced to satisfy boundary conditions, uses a graphical method for the construction of equipotentials and lines of force.

4. Experimental - We have seen that Laplace's equation can be used in the solution of a variety of physical phenomena such as electrostatics, magnetostatics, heat transfer etc., but it is also true that an experimental solution obtained in one branch of physics can be applied directly to the corresponding problem in another branch. The method consists of constructing the conductor system on a flat conducting surface and with the aid of probes determine the field potential at any point. Experimental methods can be applied to two and three dimensional fields.

In this thesis we are interested mainly in the method of Conformal Transformations and in using this method to investigate the two-dimensional fields of various conductor shapes.

1.3 Conformal Transformations

The method of conformal transformations was first used in the last century by Maxwell and others who employed a known transformation equation and worked backwards to obtain the field. In the late 1860's Schwarz [1] and Christoffel [2] developed a method of transforming the outline of a polygon in one complex plane onto the real axis of another complex plane thereby allowing a particular conductor shape to be investigated directly. This opened up the method to more practical applications and was used subsequently in electrical problems by Kirchoff [3] and Potier [4] and in hydrodynamical problems by Michell [5] and Love [6].

In 1900 Carter [7] published a paper "Note on airgap and interpolar induction", which was the first application of conformal transformations to an actual engineering problem. Page [8] in 1911 extended the method to problems containing curved lines while in the 1920's Carter [9], and Coe and Taylor [10] added elliptic functions to the already mounting repertoire of the mathematically minded engineer.

In this thesis I have tried to build up a picture of the role of conformal transformations in the solution of Laplace's equation for two-dimensional fields and show how the theory can be applied to obtain several physical properties of the field.

Chapters 2, 3 and 4 cover the theory, methods of solution and examples, and provide the information necessary in the investigation of solvable problems. The two examples chosen, illustrate the different boundary conditions that are found in electrical field theory.

Chapter 5 investigates various corner configurations which in their analysis provide examples of each of the methods of solution listed in chapter 3. They show also, the degree of mathematical manipulation often necessary in obtaining the final transformation.

Finally chapter 6 outlines two alternative methods of field determination; the relaxation method and an experimental method.

The two types of log used in this thesis, namely log and ln, both refer to Napierian logarithms.

CHAPTER 2

THEORY OF CONFORMAL TRANSFORMATIONS

2.1 Introduction to Complex Variables

The theory of conformal transformations is based on the use of functions of a complex variable and hence a short introduction to the concept of complex variables is necessary in any dissertation on this subject.

Since the product of any real number, positive or negative, by itself is positive, it is clear that no ordinary or real number has a negative square. It was therefore natural to refer to the square root of a negative number as being "imaginary". Furthermore since we can write

$$\sqrt{-x^2} = \sqrt{(-1)x^2} = x\sqrt{-1}$$

it is clear that the pure imaginary unit $\sqrt{-1}$, which is usually denoted i, can, so to speak, be made to bear the entire brunt of "imaginariness", and the square root of any negative number can be written as the product of the imaginary unit i, and a real number x.

It is often convenient when multiplying one real number x by a second positive real number y to give on geometrical grounds an operational interpretation to the product yx, and say that y is an operator which stretches x in the ratio y to 1. If the stretching operator be negative and equal to -y we naturally say that

$$-y = (-1)\,y$$

carries out two operations, the first of which is to stretch x in the ratio y to 1, while the second is a reversal of direction. This reversal of direction may itself be viewed as a rotation through 180°, so that -1 is an operator which leaves lengths unchanged but which rotates through +180°. This is shown in figure (2.1) where x is multiplied by -y to give -yx.

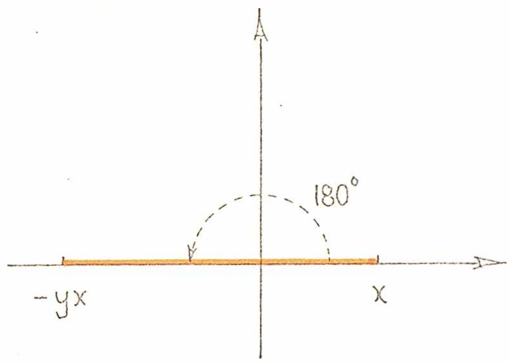


Figure 2.1 Use of -1 operator.

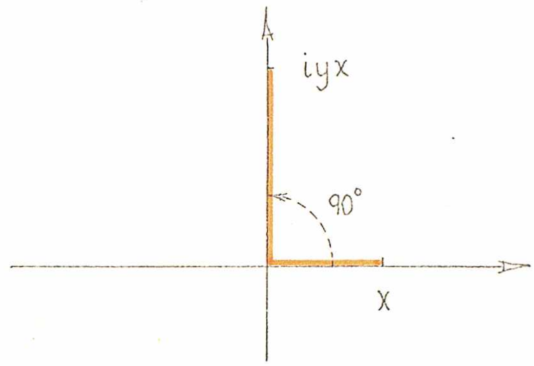


Figure 2.2 Use of i operator.

If we seek similarly to give an operational interpretation to the process of multiplying a real number x by i, we receive a useful suggestion from the fact that a double application of the operation of multiplying by i is equivalent to the operation of multiplying by -1. We therefore find it consistent with our established theory, to treat i as an operator which rotates through +90°. This can be seen in figure (2.2) where our real number x is this time operated on by $y\sqrt{-1} = i\,y$.

A mixed number x + i y which consists of the sum of a real portion x and a pure imaginary portion i y is called a "complex" number and the operational interpretation just suggested indicates the manner in which one may give a convenient geometrical interpretation to such a complex number. This is shown in figure (2.3) where the real number y is operated on by i and rotated through 90°. When complex numbers are represented in this familiar way the figure is called a Gauss-Argand diagram.

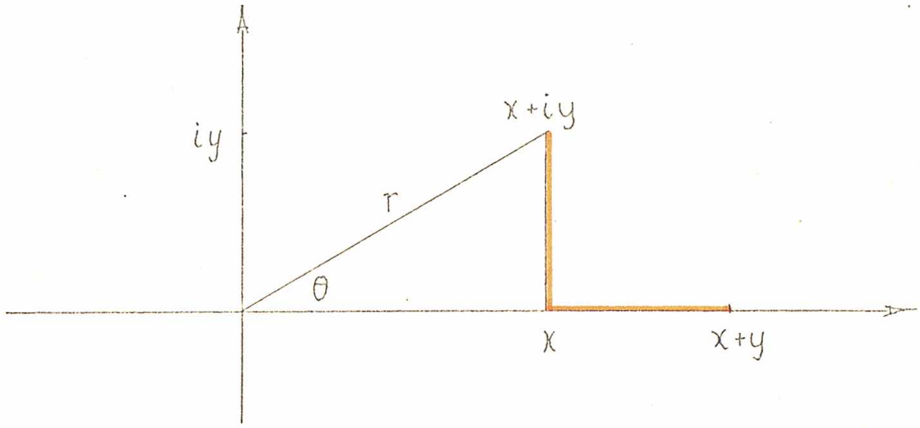


Figure 2.3 Representation of Complex Number.

It is clear from such a diagram that complex numbers add vectorially, and that two complex numbers are equal if and only if their real and imaginary parts are separately equal. It is also clear that any complex number x + i y can be written in the equivalent form r(cosθ + i sinθ) = r exp (iθ). This latter expression is called the polar form of the complex number, r and θ being known respectively as the modulus and amplitude of x + i y. This is also shown in figure (2.3).

A complex quantity whose real and pure imaginary parts are variable is called a complex variable, and may be denoted by z = x + i y or by w = u + i v where u and v are real. A functional relationship w = f(z) may exist between two such complex variables, so that to each chosen value of z there corresponds a value of w. Each of the complex quantities w and z can be represented on an Argand diagram, so that one speaks of the w-plane and the z-plane. The functional relationship sets up a correspondence between the points on the w-plane and the points on the z-plane. Thus when the variable z moves in its plane over a curve C, the corresponding point w describes a curve C' in the w-plane. One of these curves is said to be the map of the other, and in this way a functional relationship w = f(z) maps the z-plane onto the w-plane.

2.2 Analytic Functions and the Laplace Equation

We have now established w = f(z) as our functional relationship and we define the derivative as

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

This introduced a new feature since $\Delta z = \Delta x + i\Delta y$ has geometrically speaking, two degrees of freedom and can approach zero along any path as shown in figure (2.4)

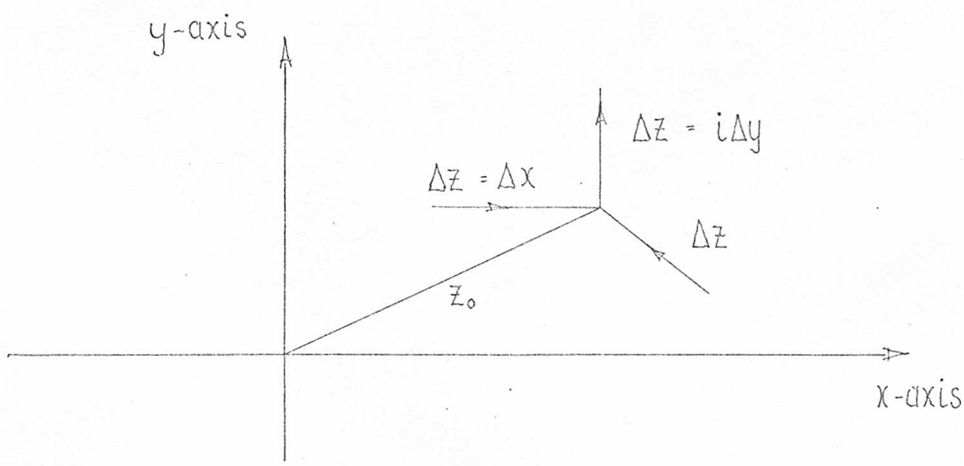


Figure 2.4 Approach to $z = z_o$

For example the point z_o in figure (2.4) may be approached along the x-axis $\Delta z = \Delta x$ or along the y-axis $\Delta z = i\Delta y$, or along any arbitrary direction Δz. The corresponding derivative dw/dz at $z = z_o$ is unaltered in either magnitude or phase. In order that the derivative of w with respect to z has a unique value for a given argument, it is necessary to demand that the limit of the above difference quotient be independent of the manner in which Δz approaches zero. If we assume for the moment that $w = u + iv = f(z)$ has a unique derivative with respect to $z = x + i y$ we have

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{dw}{dz} \cdot \frac{\partial z}{\partial x} = \frac{dw}{dz}$$

and $$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} = \frac{dw}{dz} \cdot \frac{\partial z}{\partial y} = i \frac{dw}{dz}$$

therefore $$\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} = i \left[\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right]$$

Equating real and pure imaginaries

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{------------------------} \quad (2.1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{------------------------} \quad (2.2)$$

These are known as the Cauchy-Riemann equations, and it can easily be shown that a necessary and sufficient condition that a complex function $w = u + i v$ possesses a derivative with respect to $z = x + i y$ is that the partial derivatives of the functions u and v exist, are continuous and satisfy equations (2.1) and (2.2), and that equations (2.1) and (2.2) are differentiable. A function of a complex variable which possesses a derivative at every point of a region is said to be "analytic" over that region.

Suppose now that $w = u + i v$ is, over a certain region, an analytic function of $z = x + i y$. The Cauchy-Riemann equations are then satisfied. If we differentiate equation (2.1) with respect to y we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}$$

and differentiate equation (2.2) with respect to x we get

$$\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x^2}$$

Hence
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{-----------------} \quad (2.3)$$

This partial differential equation is known as the Laplace equation in two-dimensions and it is of widespread occurrence in problems of pure and applied mathematics. This is the fundamental equation in problems of electrostatics with which this thesís is mainly concerned. By interchanging the variables when differentiating the Cauchy-Riemann equations we obtain similarly

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{-----------------} \quad (2.4)$$

so that every analytic function automatically furnishes us with a pair of real functions of two real variables each of which is a solution of Laplace's equation. The function $v(x, y)$ is called the conjugate of the function $u(x, y)$ and vice versa, and the process of obtaining solutions of the Laplace equation in this way is known as the method of conjugate functions.

From the relationship $w = f(z)$ we can separate real and imaginary parts to obtain the equations

$$u = u(x, y) \quad \text{------------------------} \quad (2.5)$$

$$v = v(x, y) \quad \text{------------------------} \quad (2.6)$$

If we take v to represent a potential and v = constant then v = v(x, y) is the equation of a curve of constant potential in the z-plane. Similarly the equation u = u (x, y) gives the corresponding lines of force in the z-plane. In this way the analytical function w = f(z) maps the w-plane onto the z-plane.

A characteristic and important feature of this mapping has not been mentioned as yet and is this: If we consider two curves in the z-plane that intersect at an angle α then it may easily be shown that if the mapping function w = f(z) be analytical, the transformed curves on the w-plane will also intersect at an angle α.

As an immediate and important application of the fact that angles are preserved in the transformation we may note that the curves u = constant in the z-plane are everywhere orthogonal to the curves v = constant, merely because the curves u = constant and v = constant in the w-plane, are straight lines which intersect orthogonally. For the curves in the z-plane to be orthogonal the product of their gradients must be -1.

Since in the relationship v = v(x, y), v is a constant

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy = 0$$

Therefore $$\frac{\partial v}{\partial x} \cdot \frac{\partial y}{\partial v} = -\left(\frac{dy}{dx}\right)_1 \quad \text{----------------} \quad (2.7)$$

Similarly from $u = u(x, y)$, u is a constant

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y} \cdot dy = 0$$

Therefore $$\frac{\partial u}{\partial x} \cdot \frac{\partial y}{\partial u} = -\left(\frac{dy}{dx}\right)_2 \qquad (2.8)$$

The two equations (2.7) and (2.8) give the gradients of the curves v = constant and u = constant in the z-plane.

Therefore $$\frac{\partial v}{\partial x} \cdot \frac{\partial y}{\partial v} \cdot \frac{\partial u}{\partial x} \cdot \frac{\partial y}{\partial u} = \left(\frac{dy}{dx}\right)_1 \cdot \left(\frac{dy}{dx}\right)_2$$

From the Cauchy-Riemann equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence $$\frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} \cdot \frac{\partial u}{\partial x} \cdot \left(-\frac{\partial x}{\partial v}\right) = \left(\frac{dy}{dx}\right)_1 \cdot \left(\frac{dy}{dx}\right)_2$$

and $$-1 = \left(\frac{dy}{dx}\right)_1 \cdot \left(\frac{dy}{dx}\right)_2$$

Thus the product of the gradients is -1 and the curves v = constant are orthogonal to the curves u = constant in the z-plane.

We have now shown that with the correct transformation equation, a family of straight orthogonal lines in the w-plane can be transformed into a corresponding family of curved orthogonal lines in the z-plane. In particular the orthogonal transformation can apply to problems in electrostatics, magnetostatics, heat, hydrodynamics and gravitational fields since the fields in some of these problems are governed by Laplace's Equation. The retention of angle during transformation is true for all angles at non-singular points, and the transformation is referred to as conformal.

2.3 The Transformation Ratio and Electrostatic Relations

When a conformal transformation is made from the z-plane to the w-plane it is a fundamental property of the theory that to any element of area in the z-plane there corresponds an exactly similar shaped area in the w-plane and that the angles at the corners of the transformed figure are unchanged, except at singular points. This conformal property is true for elements of area, not for finite areas. The relative sizes and orientations of such transformed areas depends upon the co-ordinates and the form of the transforming equation.

If we consider two points on the z-plane having separation Δz and the transformed points on the w-plane having separation Δw, then we can define the linear transformation ration λ as

$$\lambda = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz} \qquad (2.9)$$

If the transformation equation is $w = f(z)$ where $w = u + i\,v$ and $z = x + i\,y$ then

$$\frac{\partial w}{\partial x} = \frac{dw}{dz} \cdot \frac{\partial z}{\partial x}$$

But

$$\frac{\partial z}{\partial x} = \frac{\partial(x + i\,y)}{\partial x} = 1$$

Hence

$$\frac{dw}{dz} = \lambda = \frac{\partial w}{\partial x}$$

i.e.

$$\lambda = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

From the Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\lambda = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

If v is a potential function then $\frac{\partial v}{\partial y} = -Y$ and $\frac{\partial v}{\partial x} = -X$ where X and Y are components of the electric intensity parallel to the x and y axis respectively.

i.e. $\lambda = -Y - iX$

so that λ is complex

If we let $\lambda = a(\cos\theta + i\sin\theta)$, where a is the modulus of the transformation and θ its argument we see that in a conformal transformation, any element is multiplied by a and rotated counterclockwise through an angle θ to produce the corresponding element in the w-plane.

Now the resultant electric intensity or field strength R is given in magnitude as

$$R = |X + iY| = |\lambda|$$

Hence from equation (2.9)

$$R = \left|\frac{dw}{dz}\right| \qquad \text{------------------} \quad (2.10)$$

The surface density of charge on the conductor is given by

$$\sigma = -\frac{1}{4\pi} \cdot \frac{dv}{dr}$$

where dr is an element of the outward drawn normal to the conductor.

From the Cauchy-Riemann equations

$$\frac{dv}{dr} = \frac{du}{ds}$$

where ds is an element of the section of the conductor

Therefore $\sigma = -\frac{1}{4\pi}\frac{du}{ds}$ ------------------ (2.11)

If Q is the total charge on the surface of the conductor we have

$$Q = \int_1^2 \sigma \, ds \qquad \text{------------------} \quad (2.12)$$

between the appropriate limits. It must be remembered that we are dealing here with a two-dimensional conductor and that the third dimension into and out of the paper stretches to infinity so that no end effects are introduced. If we only consider a strip of unit thickness normal to the z-plane then

$$Q = -\frac{1}{4\pi}\int_1^2 \frac{du}{ds} \cdot ds$$

$$= -\frac{1}{4\pi}\left[u_2 - u_1\right]$$

hence $$Q = \frac{1}{4\pi}\left[u_1 - u_2\right] \qquad \text{(2.13)}$$

If we are dealing with a parallel plate capacitor then we will be interested in the capacitance. If the total charge is given by equation (2.13) then we can write that

$$C = \frac{Q}{V}$$

where V is the voltage difference between the two plates.

If the potential of the plates are V_1 and V_2 the capacitance is given as

$$C = \frac{(u_2 - u_1)}{4\pi\,(V_2 - V_1)} \qquad \text{(2.14)}$$

This shows very briefly how different properties of the electric field can be found from the transformation relations. The use of these properties is exemplified in the case of the parallel plate capacitor.

To summarize, we have established the transformation $z = f(w)$ which transforms the conductor in the z-plane into the w-plane and sets up a one-to-one correspondence between the z and w-planes. To change the problem into an electrical one we must now break this transformation up into two steps by introducing an intermediate t-plane. The effect of this is to provide a bridge between the purely geometrical figure in the t-plane and its electrical counterpart in the w-plane. The two steps $z = f(t)$ and $t = f(w)$ are known as the geometrical and electrical transformations respectively.

The next chapter deals with the various methods of obtaining these two transformations.

CHAPTER 3

METHODS OF SOLUTION

3.1 The Geometrical Transformation.

The following are a selection of the most widely used methods for obtaining the Geometrical Transformation $z = f(t)$. Most of them involve transforming the outline of the conductor in the z-plane onto the real axis of the t-plane.

3.1.1 Solution from a known transformation

This is the original method used by Maxwell [11] and Thomson [12] where we start with a given transformation equation and work back to find the shape of the curves in the z-plane. This method suffers from the obvious disadvantage that there is no rule given for determining the proper transformation for any particular problem. As 'J.J. Thomson' put it: "Success in using these methods depends chiefly upon good fortune in guessing the suitable transformation" [12]. There are many relations $z = f(w)$ that have been investigated and perhaps to investigate one would best explain the method.

Consider the conformal transformation

$$z = k \cosh w \qquad (3.1)$$

where z and w are complex and k is a real constant.

If $z = x + i y$ and $w = u + i v$ then

$$x + i y = k \cosh (u + i v)$$

From what was said in the last chapter we require to find the shape of the curves v = constant and u = constant in the z-plane.

Expanding the cosh term we get

$$x + i y = k (\cosh u \cos v + i \sinh u \sin v)$$

separating real and imaginary parts gives

$$x = k \cosh u \cos v \qquad \text{------------} \quad (3.2)$$

$$y = k \sinh u \sin v \qquad \text{------------} \quad (3.3)$$

Squaring and rearranging gives

$$\cos^2 v = \frac{x^2}{k^2 \cosh^2 u} \qquad \text{------------} \quad (3.4)$$

$$\sin^2 v = \frac{y^2}{k^2 \sinh^2 u} \qquad \text{------------} \quad (3.5)$$

adding

$$\sin^2 v + \cos^2 v = \frac{y^2}{k^2 \sinh^2 u} + \frac{x^2}{k^2 \cosh^2 u} = 1$$

$$\text{------------} \quad (3.6)$$

This is the equation of an ellipse in the z-plane and hence the curves u = constant are a family of confocal ellipses.

If we rewrite (3.4) and (3.5) as

$$\cosh^2 u = \frac{x^2}{k^2 \cos^2 v} \qquad \text{------------} \quad (3.7)$$

$$\sinh^2 u = \frac{y^2}{k^2 \sin^2 v} \qquad \text{------------} \quad (3.8)$$

and subtract, we get

$$\cosh^2 u - \sinh^2 u = \frac{x^2}{k^2 \cos^2 v} - \frac{y^2}{k^2 \sin^2 v} = 1$$

$$\text{------------} \quad (3.9)$$

This is the equation of a hyperbola and thus the curves v = constant are a family of confocal hyperbolas in the z-plane. Figure (3.1) shows the w and z planes with the respective v = constant and u=constant lines.

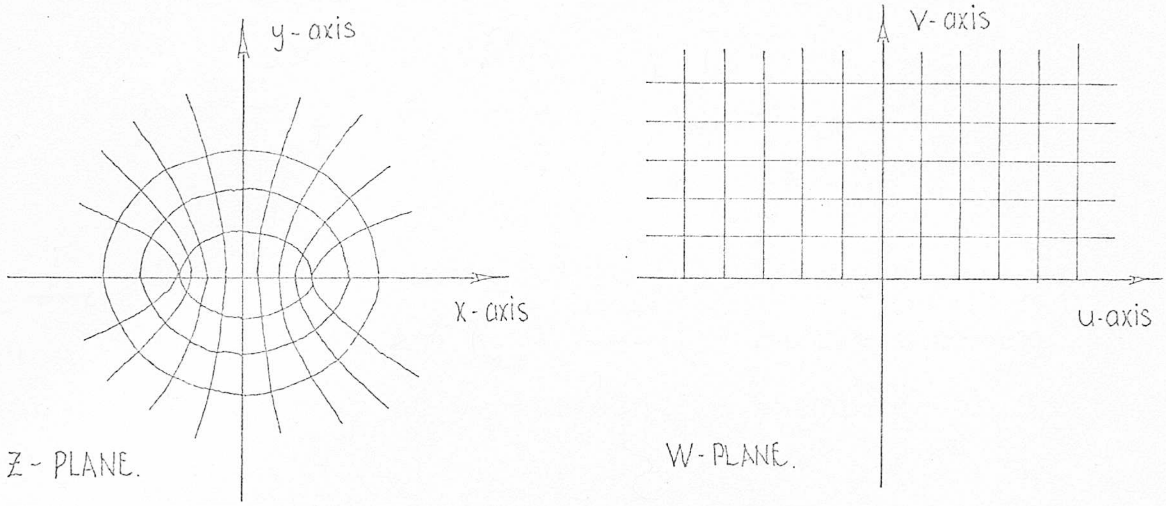


Figure (3.1) The transformation $z = k \cosh w$.

If we assume that the conductor in the z-plane is an ellipse, then the outer ellipses will represent equipotentials and the hyperbolas will represent lines of force, i.e. u = constant will be equipotentials and v = constant will be lines of force. Therefore we can say that the relation $z = k \cosh w$ transforms the lines of equipotential and lines of force of an infinitely long conductor, whose cross section in the z-plane is an ellipse, into a rectangular grid in the w-plane.

3.1.2 The Schwarz-Christoffel transformation

This is probably the most commonly used method in conformal transformations as it involves the direct transformation of the conductor which is the perimeter of a polygon in the z-plane on to the real axis of the t-plane. The field inside the conductor is transformed into the upper half of the t-plane while the field outside becomes the lower half. The transformation equation is given as

$$\frac{dz}{dt} = A(t - t_1)^{\frac{\alpha_1}{\pi} - 1} \cdot (t - t_2)^{\frac{\alpha_2}{\pi} - 1} \cdots (t - t_n)^{\frac{\alpha_n}{\pi} - 1}$$

where t_1 t_2 ---- t_n are values of t on the real axis of the t-plane which correspond to the corners of the polygon in the z-plane and α_1 α_2 ---- α_n are the internal angles of the polygon progressing in a counter clockwise direction.

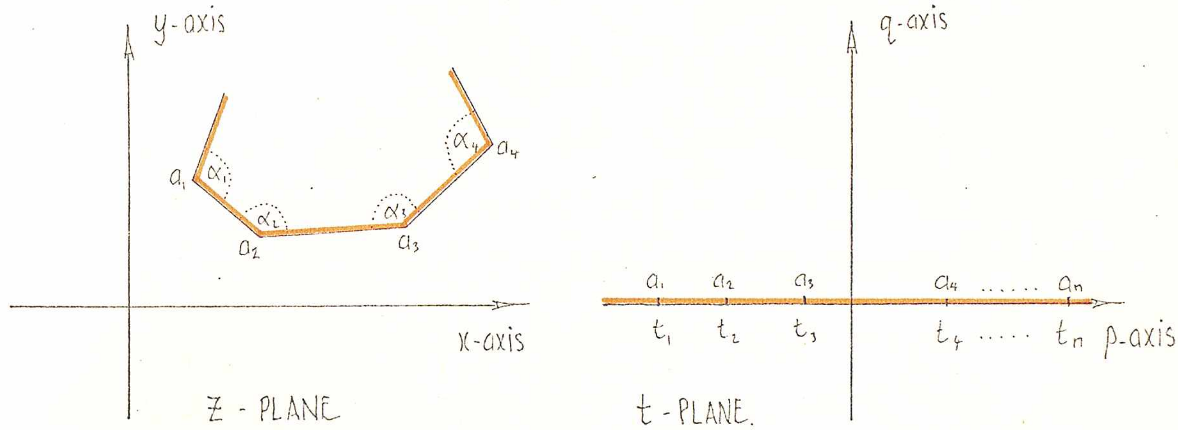


Figure 3.2. Transformation of polygon on to real axis of t-plane.

A is a complex constant. Figure (3.2) shows the outline of the polygon in the z-plane and the transformed perimeter in the t-plane.

The equation is usually written in the form

$$\frac{dz}{dt} = A \prod_{r=1}^{n} (t - t_r)^{\frac{\alpha_r}{\pi} - 1} \qquad \text{------------- (3.10)}$$

and when integrated we get the transformation equation

$$z = A \int \prod_{r=1}^{n} (t - t_r)^{\frac{\alpha_r}{\pi} - 1} \, dt + B \qquad \text{-------- (3.11)}$$

where B is another complex constant which with A has to be determined by boundary conditions.

The modulus of the constant A determines the size of the polygon and the argument of A determines the orientation. The location of the polygon is determined by the constant B.

When we wish to transform any given polygon in the z-plane on to the real axis of the t-plane we have the values α_1 α_2 .... α_n given. As regards the values t_1 t_2 --- t_n some may be arbitrarily assumed while others will have to be determined from the dimensions of the polygon. Whatever the values of t_1 t_2 ---- t_n the transformation equation (3.11) will transform the real axis of the t-plane into a polygon whose internal angles have the required values α_1 α_2 ---- α_n. In order that this polygon be similar to the given one we require n-3 conditions to be satisfied; hence as regards the n quantities t_1 t_2 --- t_n the values of 3 of them may be arbitrarily assumed while the remaining n-3 must be determined by the dimensions of the polygon in the z-plane. If one of the values of t chosen happens to be infinity then the factor containing it may be ignored.

3.1.3 Complex geometrical inversion

This is an indirect method of obtaining the Geometrical Transformation, producing new problems, from the solution of a previous system of conductors.

The general equation for inversion from the z-plane to the z_1 plane is

$$z \cdot z_1 = \text{constant} \qquad \text{---------------} \quad (3.12)$$

where the constant is real and equal to the square of the radius of inversion, z is the transformation equation between the z and t planes and z_1 is the corresponding transformation equation for the new problem. If we wish to invert about a point which is not the origin of the given z-plane then we have to move the origin of the z-plane to that point before inversion.

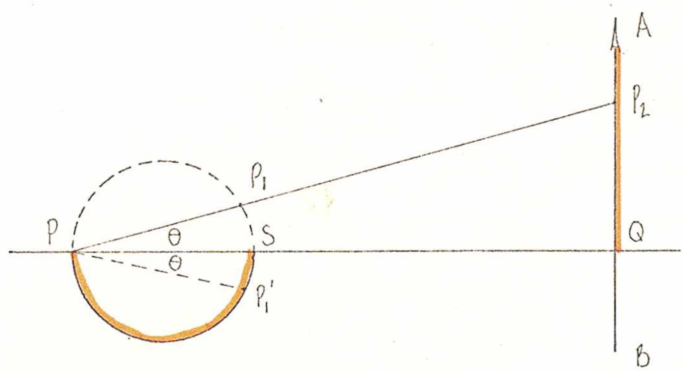


Figure 3.3. Complex inversion of AQ into semicircle PS.

The technique of inversion can best be explained by referring to an example. Figure (3.3) shows two points P_1 and P_2 in the z-plane lying on a line through the point P. The point P_1' is the image of P_1 in the horizontal line PQ. Suppose the points P_1 and P_2 are connected by the equation

$$PP_1 \cdot PP_2 = h^2 \qquad \text{---------------} \quad (3.13)$$

where h is a real constant. Then as P_2 moves along the vertical line AB where A is at infinity, the locus of the point P_1 would move round the dotted semi-circle, assuming both P_1 and P_2 were real numbers. This is geometrical inversion with the centre of inversion at P and a radius of inversion of h, and shows how the semi-infinite straight line AQ inverts into a semi-circle.

If however the two points P_1 and P_2 are complex we have

$$P_1 = \frac{h^2}{P_2}$$

and if $P_2 = Re^{i\theta}$, then

$$P_1 = \frac{h^2}{R} e^{-i\theta}$$

and P_1 now becomes the reflection point P_1' as shown in figure (3.3). Hence the semi-circle is reflected in the PQ plane and we can say that the straight line AQ will be inverted into a semi-circle travelling anti-clockwise from P. The complete inversion is referred to as the complex geometrical inversion.

3.1.4 Richmond's Method

This method is used for certain cases when the z-plane figure is a triangle or quadrilateral some of whose sides are arcs of a circle. The figure is transformed into a rectangle in the t-plane which can be investigated in the usual manner.

Consider the transformation

$$t = \log z \qquad (3.14)$$

where $t = p + i\,q$ and $z = x + i\,y = r \exp i\theta$, then

$$p + i\,q = \log r + i\theta$$

Therefore

$$p = \log r \qquad (3.15)$$

$$q = \theta \qquad (3.16)$$

When p is constant log r = constant and the curves in the z-plane corresponding to p = constant in the t-plane are circles. If q is held constant then θ = constant and the lines in the z-plane corresponding to q = constant are radii from the origin of the z-plane. Thus a rectangular mesh in the t-plane transforms into an orthogonal mesh of concentric circles and radii in the z-plane. This is shown in figure (3.4).

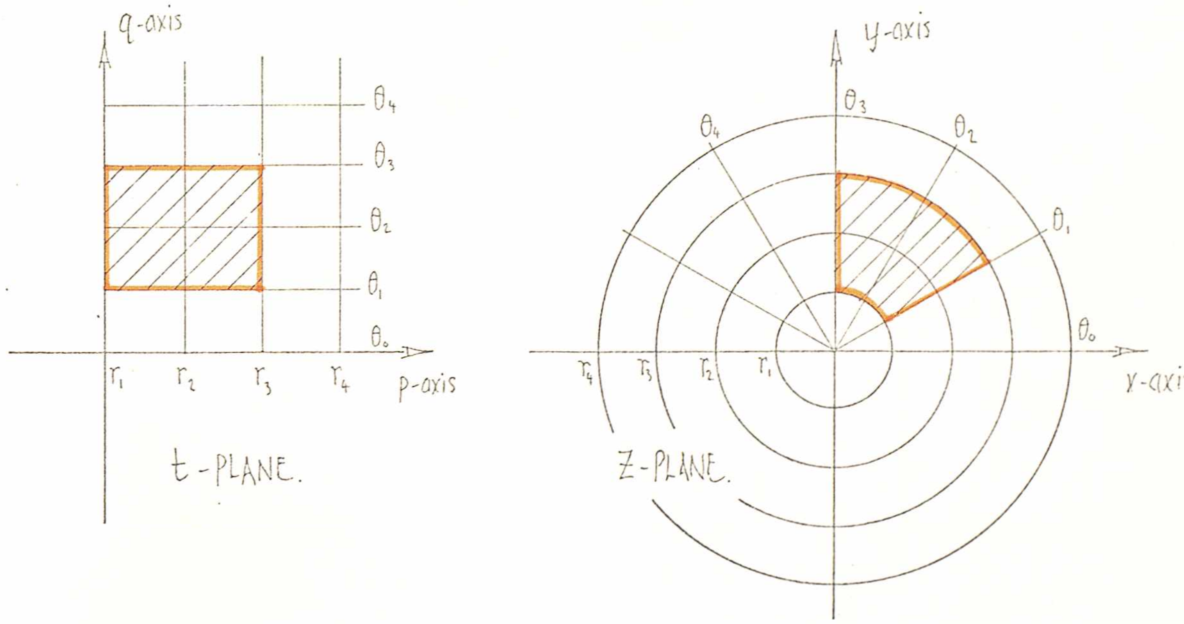


Figure 3.4 Transformation $t = \log z$.

If $r < 1$ then p is negative, and therefore the area inside the circle of unit radius in the z-plane corresponds to the negative upper quadrant of the t-plane, and the area outside this circle corresponds to the positive upper quadrant.

If the conductor to be investigated is formed by a combination of circular arcs and corresponding radii, then its transformation into a rectangle by this method can often simplify the investigation.

3.1.5 Selection

This method involves using part of the conductor or field of a known problem to obtain a new problem which cannot be solved directly. Figure (3.6a) shows the conductor ABC in the z-plane with the field of interest external to the conductor. By the theory of curvilinear triangles, which will be explained in the next section the outline of the conductor is transformed on to the real axis of the t-plane with the field external to the conductor becoming the upper half of the t-plane. This is shown in figure (3.6b).

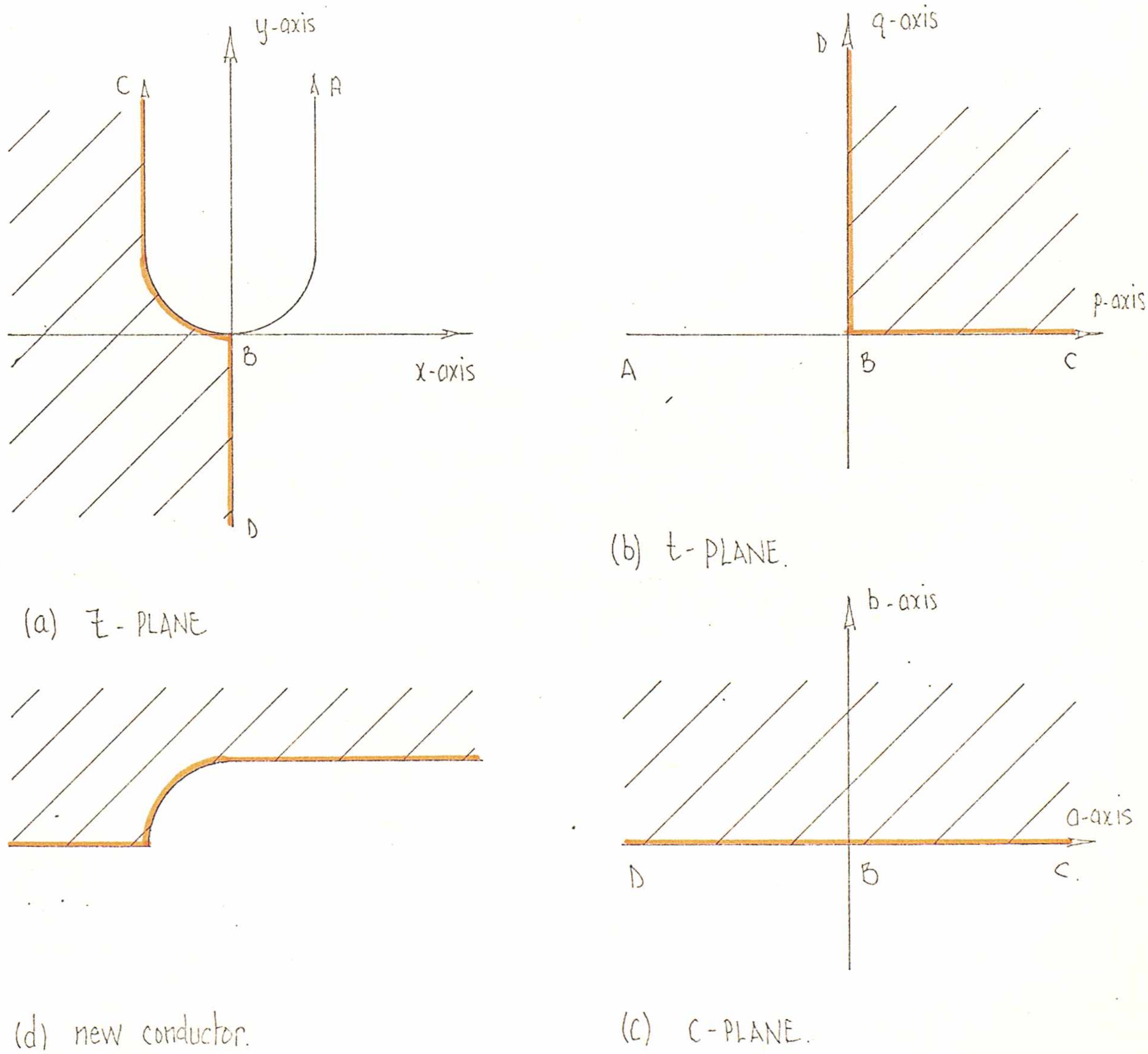


Figure 3.6. New conductor shape by selection.

If we select the quadrant CBD in the t-plane and transform the outline of the polygon CBD on to the real axis of the c-plane then we will have effectively transformed the figure CBD in the z-plane on to the real axis of the c-plane. The shaded areas of figures (3.6a), (3.6b) and (3.6c) show the transformed fields in each case. Thus we have found the solution of a new problem by selecting part of the field of a known one. The new conductor shape is shown in figure (3.6d).

3.1.6 Method of Curvilinear Triangles

This method will find the transformation for a conductor whose trace in the z-plane is formed from three intersecting circles. Two of these circles may have infinite radii, so that the resulting figure can have two straight sides. The method is to find the transformation from the z-plane to the c-plane using the integral solution of the hypergeometric series. This series is

$$y = \frac{\alpha.\beta.x}{1!\,\gamma} + \frac{\alpha(\alpha+1).\beta(\beta+1).x^2}{2!\,\gamma(\gamma+1)} + \dots \qquad (3.17)$$

This is denoted by $y = F(\alpha,\beta,\gamma,x)$.

If α and β are negative the series is finite, if not then it is infinite, assuming γ is not negative.

The differential equation of $F(\alpha,\beta,\gamma,x)$ is

$$x(1-x).\frac{d^2y}{dx^2} + \left[\gamma - (\alpha+\beta+1)x\right].\frac{dy}{dx} - \alpha.\beta.\gamma = 0 \qquad (3.18)$$

The integral solution of this differential equation is

$$y = A \int_0^1 v^{\beta-1}.(1 - v)^{\gamma-\beta-1}.(1 - x.v)^{-\alpha}\, dv \qquad \text{------------ (3.19)}$$

where A is a real constant and β is positive and $\gamma > \beta$. There are 23 other integral solutions which can be obtained from equation (3.19) by manipulation.

It is shown by Forsyth [13] that the elements α, β and γ of the hypergeometric integral are connected with the internal or external angles of a curvilinear triangle in the z-plane as follows. If the angles at the corners of the triangle are $\lambda\pi$, $\mu\pi$ and $\nu\pi$, then

$$\begin{aligned} \lambda^2 &= (1 - \gamma)^2 \\ \mu^2 &= (\alpha - \beta)^2 \\ \nu^2 &= (\gamma - \alpha - \beta)^2 \end{aligned} \qquad \text{------------ (3.20)}$$

From these values we can obtain values of α, β and γ. These values must satisfy the appropriate conditions such that β must be positive and that $\gamma > \beta$, or whatever the conditions are for the version of the integral equation to be used. We then pick two integrals from the 24 possible solutions and put z for y and let x = c, where $c = k^2$ and k is the modulus of the elliptic functions. We are therefore transforming from the z-plane to the c-plane. We do this because the integral equation for z will be elliptic in each case. We obtain two equations of the form

$$z = A \int_0^1 v^{\beta-1}.(1 - v)^{\gamma-\beta-1}.(1 - cv)^{-\alpha}.dv$$

------------ (3.21)

This type of equation is solved by putting $v = sn^2 u$ and the limits will change to k and o. Thus we obtain two values of z of the form $z_1 = f_1(u,k)$ and $z_2 = f_2(u,k)$. To complete the transformation a bilinear equation is used and the final transformation is of the form

$$z = \frac{A_1 z_1 + A_2 z_2}{B_1 z_1 + B_2 z_2} \qquad \text{----------------} \quad (3.22)$$

where A_1 A_2 B_1 and B_2 are constants to be determined. Of these four constants only three are independent and these can be found from the values of z and c at the three corners of the triangle in the z-plane.

Thus the completed transformation which will transform the boundary of the conductor in the z-plane onto the real axis of the c-plane is obtained. The method is limited to figures having three corners and is further limited in that the integral equation (3.21) can only be evaluated when it simplifies to become an integral involving the elliptic functions raised to whole number powers.

3.2 The Electrical Transformation

The object of the electrical transformation is two-fold. Firstly it is to change the problem from a purely geometrical transformation into an electrical one by introducing an electrical w-plane where $w = u + i\,v$. Secondly, it is to establish the correct boundary conditions for the conductor.

To fulfill the first object it is necessary to transform part of the w-plane on to the upper half of the t-plane. In simple problems where the conductor in the z-plane is represented by a single infinite plane, the perimeter of the conductor is transformed on to the real axis of the t-plane and the field external to the conductor is made the upper half of the t-plane.

If the potential of the conductor in the z plane is v_o then the electrical transformation must make the horizontal line $w = u + iv_o$ in the w-plane transform on to the real axis of the t-plane. In this way the potential of the conductor has obtained its correct value relative to the zero axis of the electrical w-plane. This is illustrated in figure (3.7) which shows the t-plane and the w-plane and the appropriate parts of each.

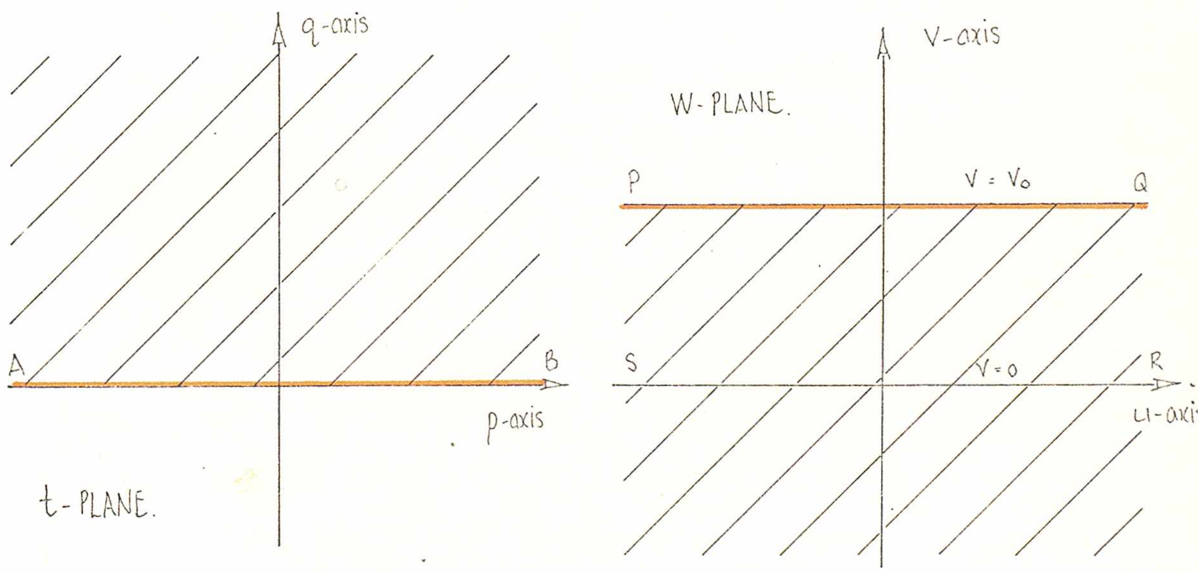


Figure 3.7. Electrical Transformation from w-plane to t-plane.

All horizontal lines drawn in the w-plane represent equipotentials and those lying between PQ and SR in figure (3.7) represent potentials between v_o and zero. Thus if we require the field between the conductor at v_o and a conductor at zero potential in the z-plane we must transform the rectangular area PQRS in the w-plane on to part of the upper half of the t-plane.

All vertical lines in the w-plane emanating from PQ downwards represent lines of force emanating from the conductor in the z-plane thus completing the electrical transformation. The rectangle in the w-plane may be finite or infinite depending on the conductor system, as explained later.

The second requirement of the electrical transformation is to set up the correct boundary conditions of the problem. At the conductor itself this is easily done as above by transforming indirectly the perimeter of the conductor on to the horizontal line $w = u + i\,v_0$ on the w-plane. The other boundary concerns the terminus of the lines of force. Now in the case of a single isolated continuous conductor in the z-plane the lines of force issuing from it must terminate on the "point at infinity".

The boundary condition used in conformal transformation is to regard infinity as a point. Thus the lines of force must be regarded as terminating on a point conductor at infinity. If we regard the conductor in the z-plane to be positively charged, as is usual, then the point at infinity will have a potential of $-\infty$. The lines of force emanating from the conductor in the z-plane will have to pass through a surface of zero potential before continuing to $-\infty$. This is illustrated in figure (3.7) where vertical lines, representing lines of force, travel downwards from $v = v_0$ through $v = 0$ to $v = -\infty$. In the case of a conductor extending to infinity, we must imagine that there is a small infinitesimal break between the point representing the corner of the conductor at ∞ and the point whose potential is $-\infty$ because the surface of zero potential must pass through this break. Thus to solve the problem of a single conductor we must transform the whole of the lower half of the w-plane plus the upper half up to the line $v = v_0$ on to the upper half of the t-plane. This gives us the electrical transformation $w = f(t)$.

In the case where there are two or more conductors in the z-plane at different potentials, the electrical condition changes. As in the case of the single conductor, the Schwarz-Christoffel equation will transform the outline of the conductors in the z-plane on to the real axis of the t-plane so that different parts of this axis represents different potentials. On introducing the electrical w-plane we must transform the appropriate potential lines on to the real axis of the t-plane.

For example, consider we have a two conductor system in the z-plane and we transform one on to the negative half and the other on to the positive half of the real axis of the t-plane. This is shown in figure (3.8a).

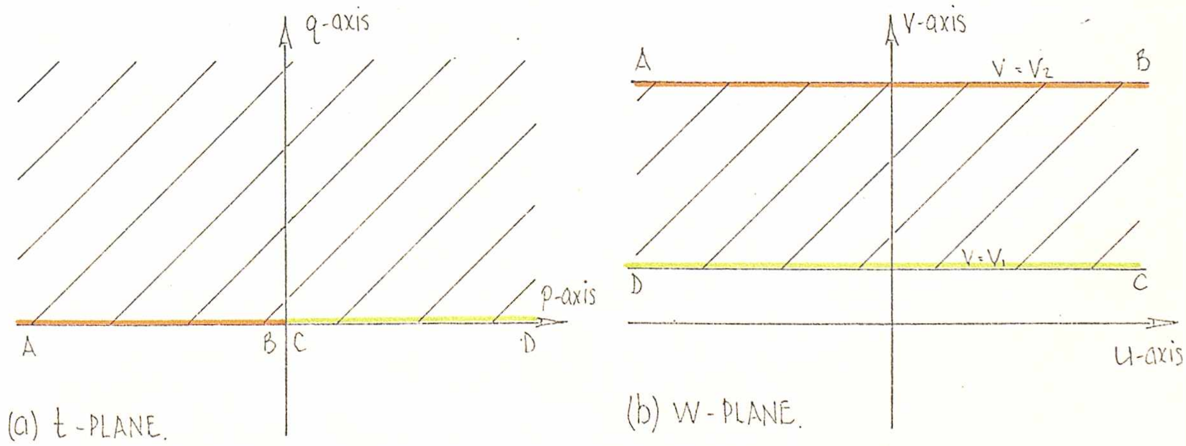


Figure 3.8. Transformation of two-conductor system.

The two semi-infinite lines AB and CD are at potentials v_2 and v_1 respectively. The electrical transformation transforms the lines $w = u + i v_2$ and $w = u + i v_1$ in the w-plane, on to AB and CD respectively in the t-plane thereby maintaining the correct relative potential values of the conductors. As regards the boundary conditions we do not need to introduce another conductor at infinity since the lines of force emanating from one conductor will fall on the other and not go to infinity. Hence the field we are interested in extends, in the w-plane, from the horizontal line AB, representing the conductor at potential v_2, to the horizontal line CD representing the conductor at potential v_1. This is shown in figure (3.8b).

Having obtained the electrical transformation $w = f(t)$ we can now use it together with the geometrical transformation $z = f(t)$ to eliminate the intermediate t-plane and establish the direct transformation $z = f(w)$.

In Chapter 4 we will see two examples of the electrical transformation being employed in particular conductor systems. In the Lee's wall problem we have an example of an isolated conductor where the lines of force terminate at a "point at infinity". In the capacitor problem we have two conductors at different potentials and hence all lines of force go from one to the other.

CHAPTER 4

THE LEES' WALL AND CAPACITOR PROBLEMS

THE LEES' WALL AND CAPACITOR PROBLEMS

In this chapter we will investigate two problems using the Schwarz-Christoffel Transformation and show how various field properties can be obtained from the basic theory. The first problem is an example of the single conductor system where the lines of force are assumed to terminate on a point object at infinity. The second problem involves two conductors where the lines of force of one all terminate on the other. These problems illustrate the different electrical boundary conditions referred to in Chapter 3.

4.1 Lees' Wall Problems

This is a series of problems investigated by Lees [14] in 1915 to find the electrostatic field near a wall or series of walls due to the potential gradient of the earth's atmosphere. The walls and ground are regarded as one capacitor plate, the other plate being at infinity. The system is thus equivalent to a polygon in the z-plane. In our analysis we will regard the wall and ground as being at some arbitrary potential V_o and investigate the field of diminishing potential above this.

Figure (4.1a) shows the ground as the x-axis of the z-plane and a simple vertical wall extending up the y-axis to a height h. The perimeter of the polygon system is thus PQRST with P and T at infinity. Since P and T meet at infinity but travel in opposite directions, the angle of contact is π. The angles at Q and S are $\frac{\pi}{2}$ and at R

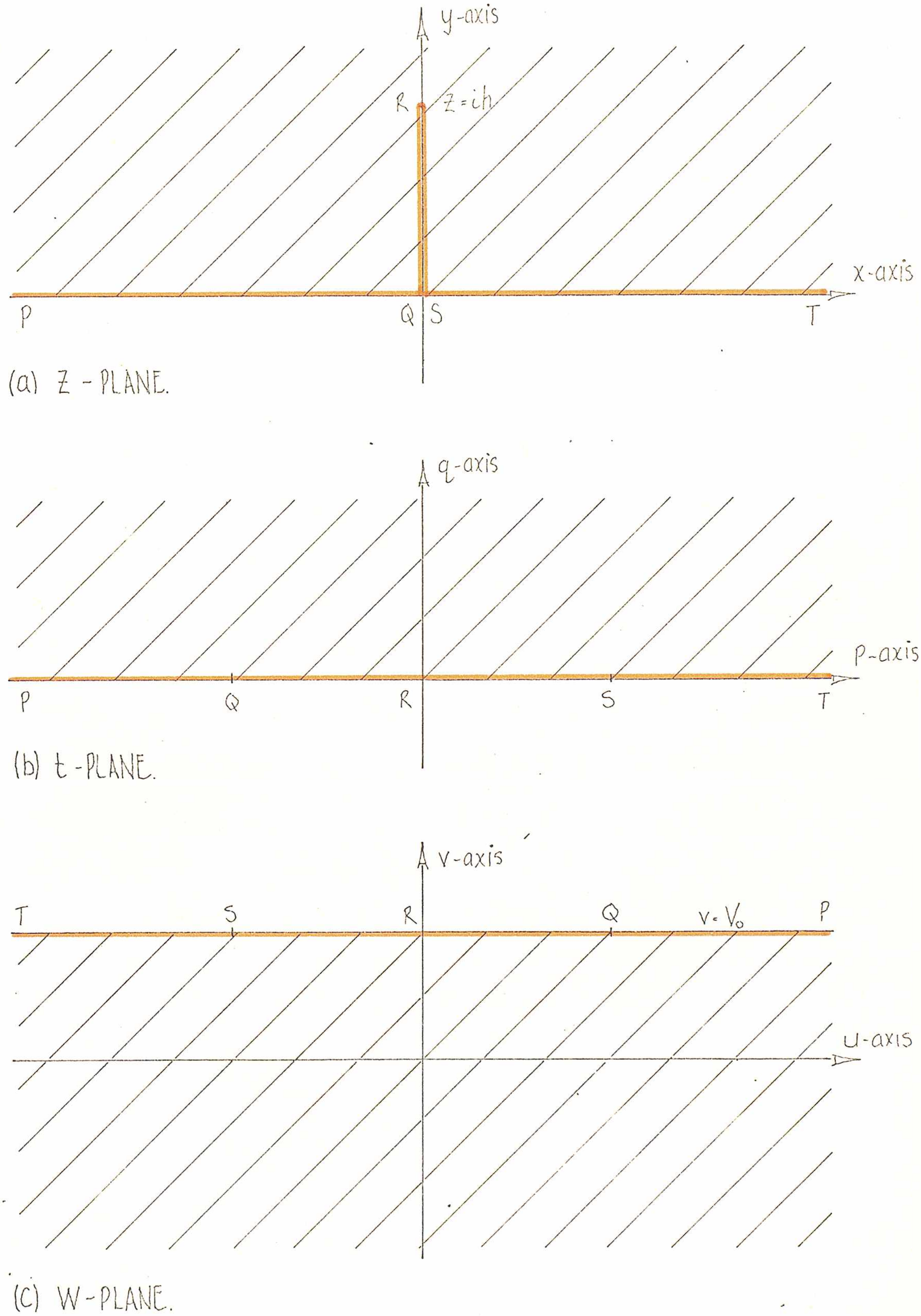


Figure 4.1 Transformation of Lee's Wall.

the angle is 2π.

We require to transform the perimeter PQRST of the polygon on to the real axis of the intermediate t-plane and find the geometrical transformation $z = f(t)$. To do this we use the Schwarz-Christoffel equation:

$$\frac{dz}{dt} = A(t - t_1)^{\frac{\alpha_1}{\pi}-1} (t - t_2)^{\frac{\alpha_2}{\pi}-1} \text{ --------- } (t - t_n)^{\frac{\alpha_n}{\pi}-1} \qquad \text{--------------- (4.1)}$$

where $t_1 t_2$ ---- t_n are the values on the real axis of the t-plane that correspond to the corners of the polygon on the z-plane, α_1 α_2 ----- α_n are the internal angles of the polygon and A is a constant.

If we let $t = \pm\infty$ when $z = \pm\infty$ we are left with three corners at Q, R and S. For simplicity we let Q correspond to $t = -1$, R correspond to $t = 0$ and S correspond to $t = +1$.

The mapping table thus becomes

Point Q	t_1	$= -1$	$\alpha_1 = \frac{\pi}{2}$
R	t_2	$= 0$	$\alpha_2 = 2\pi$
S	t_3	$= 1$	$\alpha_3 = \frac{\pi}{2}$

Substituting these values into equation (4.1) we get

$$\frac{dz}{dt} = A\,(t+1)^{-\frac{1}{2}}\,(t-0)^{1}\,(t-1)^{-\frac{1}{2}}$$

$$= A\,\frac{t}{\sqrt{(t+1)(t-1)}} \qquad \text{--------------- (4.2)}$$

Therefore $z = A\displaystyle\int \frac{t}{\sqrt{t^2-1}}\,dt$

Hence $\quad z = A\sqrt{t^2 - 1} + B \qquad$ --------------- (4.3) where B is an integration constant. To find the values of the two constants we must introduce boundary conditions into equation (4.3).

1. At the point R, $z = ih$ and $t = 0$ hence

$$ih = A\sqrt{0 - 1} + B$$

therefore $A = iB + h$ ————— (4.4)

2. At the point Q, $t = -1$ and $z = 0$ hence

$$0 = A\sqrt{1 - 1} + B$$

therefore $B = 0$ ————— (4.5)

Hence from equation (4.4)

$$A = h.$$

Substituting the values of A and B into equation (4.3) we get

$$z = h\sqrt{t^2 - 1} \qquad \text{————— (4.6)}$$

This is the geometrical transformation equation $z = f(t)$ which transforms the outline of the polygon in the z-plane on to the real axis of the t-plane. The field inside the polygon, i.e. the upper half of the z-plane, becomes the upper half of the t-plane. This is shown in figures (4.1a) and (4.1b) where we see the outline on the real axis of the t-plane. The relevant fields are shaded.

We must now introduce the electrical w-plane and find the electrical transformation $w = f(t)$ in order to change the problem from a geometrical case into an electrical one with the relevant boundary conditions.

If the potential of the ground and wall is V_o then we must transform the horizontal line $w = u + iV_o$ in the w-plane on to the real axis of the t-plane and the field below the line $w = u + iV_o$ into the upper half of the t-plane. The w-plane is shown in figure (4.1c) with the relevant field shaded. It will be seen from figures (4.1b) and (4.1c) that the outline of the polygon PQRST has to be rotated through 180° in order to obtain the correct field orientation. In the original Lee's Wall Problem this was unnecessary since the wall and ground were at zero potential with potential increasing upwards from the ground. Hence the upper half of the w-plane was transformed on to the upper half of the t-plane, and no rotation was required. The required transformation equation can be seen to be

$$w = -t + iV_o \qquad \text{------------ (4.7)}$$

We must now bring equations (4.6) and (4.7) together to eliminate the intermediate t-plane and establish the direct transformation z = f(w) between the z and w planes. We have from equation (4.6)

$$z = h\sqrt{t^2 - 1}$$

from equation (4.7) we get

$$t = i\,V_o - w$$

therefore

$$z = h\sqrt{(i\,V_o - w)^2 - 1} \qquad \text{------------- (4.8)}$$

This is the direct transformation z = f(w). But z = x + i y and w = u + iv, hence if we expand equation (4.8) we get

$$x + iy = h\sqrt{(i\,V_o - u - i\,v)^2 - 1}$$

$$= h\sqrt{-V_o - 2\,i\,u\,V_o + 2vV_o + u^2 + 2iuv - v^2 - 1}$$

Squaring both sides gives

$$x^2 + 2ixy - y^2 = h^2\,(-V_o - 2iuV_o + 2vV_o + u^2 + 2iuv - v^2 - 1)$$

Separating real and imaginary parts gives

$$x^2 - y^2 = h^2(-V_o + 2vV_o + u^2 - v^2 - 1) \quad \text{------} \quad (4.9)$$

$$2xy = h^2(-2uV_o + 2uv) \quad \text{----------} \quad (4.10)$$

From these equations we can see that it would be difficult to obtain x and y directly as functions of u and v only, therefore we will use these two equations to obtain firstly y as a function of x and v in order to find the shapes of the equipotentials, and secondly x as a function of y and u, in order to find the lines of force. To simplify the mathematics we will let h equal 1 unit of height and V_o equal 1 unit of potential difference. Hence from equation (4.10) we have

$$2\,xy = -\,2u + 2uv$$

$$xy = u\,(v-1)$$

Therefore $$u = \frac{xy}{v-1} \quad \text{----------} \quad (4.11)$$

Substituting this into equation (4.9) gives

$$x^2-y^2 = -1 + 2v + \frac{x^2y^2}{(v-1)^2} - v^2 - 1$$

$$y^2 + \frac{x^2\,y^2}{(v-1)^2} = x^2 + 2 - 2v + v^2$$

Therefore $$y^2 = \frac{(v-1)^2.\,[x^2 + 2 - 2v + v^2]}{x^2 + (v-1)^2}$$

and $$y = \sqrt{(v-1)^2.\,\frac{[x^2 + (v-1)^2 + 1]}{x^2 + (v-1)^2}} \quad \text{----} \quad (4.12)$$

If we rearrange equation (4.11) we get

$$v = \frac{xy + u}{u} \quad \text{----------} \quad (4.13)$$

Substituting this into equation (4.9) now gives

$$x^2 - y^2 = -1 + \frac{2xy}{u} + 2 + u^2 - \left(\frac{xy + u}{u}\right)^2 - 1$$

$$= \frac{2xy}{u} + u^2 - \frac{x^2y^2}{u^2} - \frac{2xy}{u} - 1$$

Thus $x^2 + \frac{x^2y^2}{u^2} = u^2 + y^2 - 1$

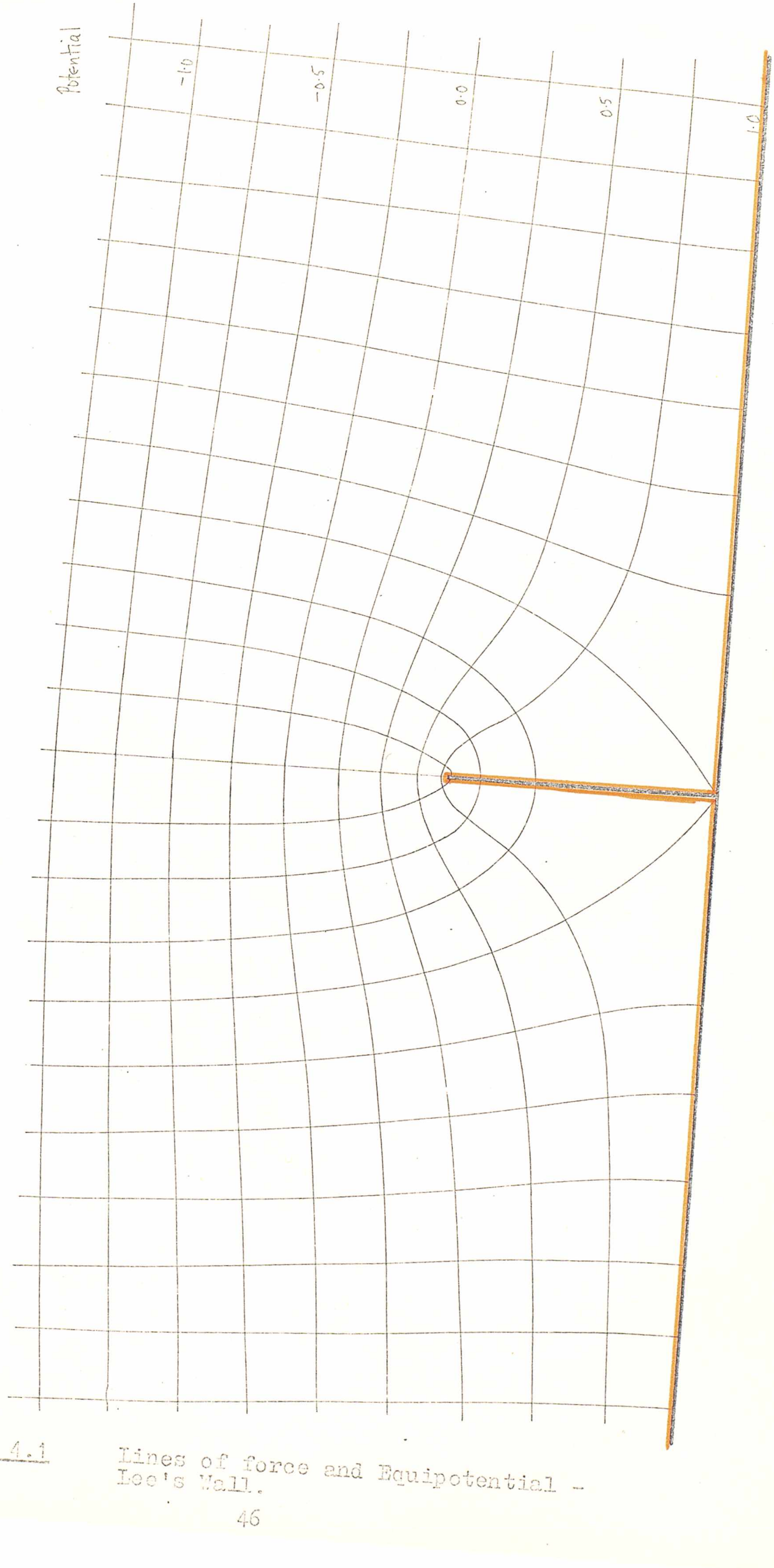
Therefore $x^2 = \frac{u^2(u^2 + y^2 - 1)}{u^2 + y^2}$

and $$x = \sqrt{\frac{u^2\ (u^2 + y^2 - 1)}{u^2 + y^2}} \quad \text{---------------} \quad (4.14)$$

Equations (4.12) and (4.14) will give us the equipotentials and lines of force in the upper half of the z-plane. By substituting different values of x for a given value of v in equation (4.12) we can find the corresponding y co-ordinates and hence plot that equipotential in the z-plane. Similarly by substituting different values of y for a given value of u in equation (4.14) we can find the corresponding x coordinates and plot that line of force.

A computer program was written to calculate the values of x and y from equations (4.12) and (4.14) for equipotentials from 0.75 to -1.25 volts in steps of 0.25 and for the lines of force over the relevant area. A plot of these values can be seen in Graph 4.1, where the effect of the wall on the potential field is seen by the curvature of the equipotentials and lines of force.

Having obtained the direct transformation equation z = f(w) we can now use it to calculate various field



Graph 4.1 Lines of force and Equipotential - Lee's Wall.

properties referred to in Chapter 2. The field strength R was given as

$$R = \left| \frac{dw}{dz} \right| \qquad \text{------------} \quad (4.15)$$

But from equation (4.8)

$$z = h \sqrt{(iV_o - w)^2 - 1}$$

Therefore $\dfrac{z^2}{h^2} + 1 = (i V_o - w)^2$

and

$$w = i V_o - \sqrt{\frac{z^2}{h^2} + 1}$$

Hence

$$\frac{dw}{dz} = -\frac{1}{2}\left(\frac{z^2}{h^2} + 1\right)^{-\frac{1}{2}} \frac{2z}{h^2}$$

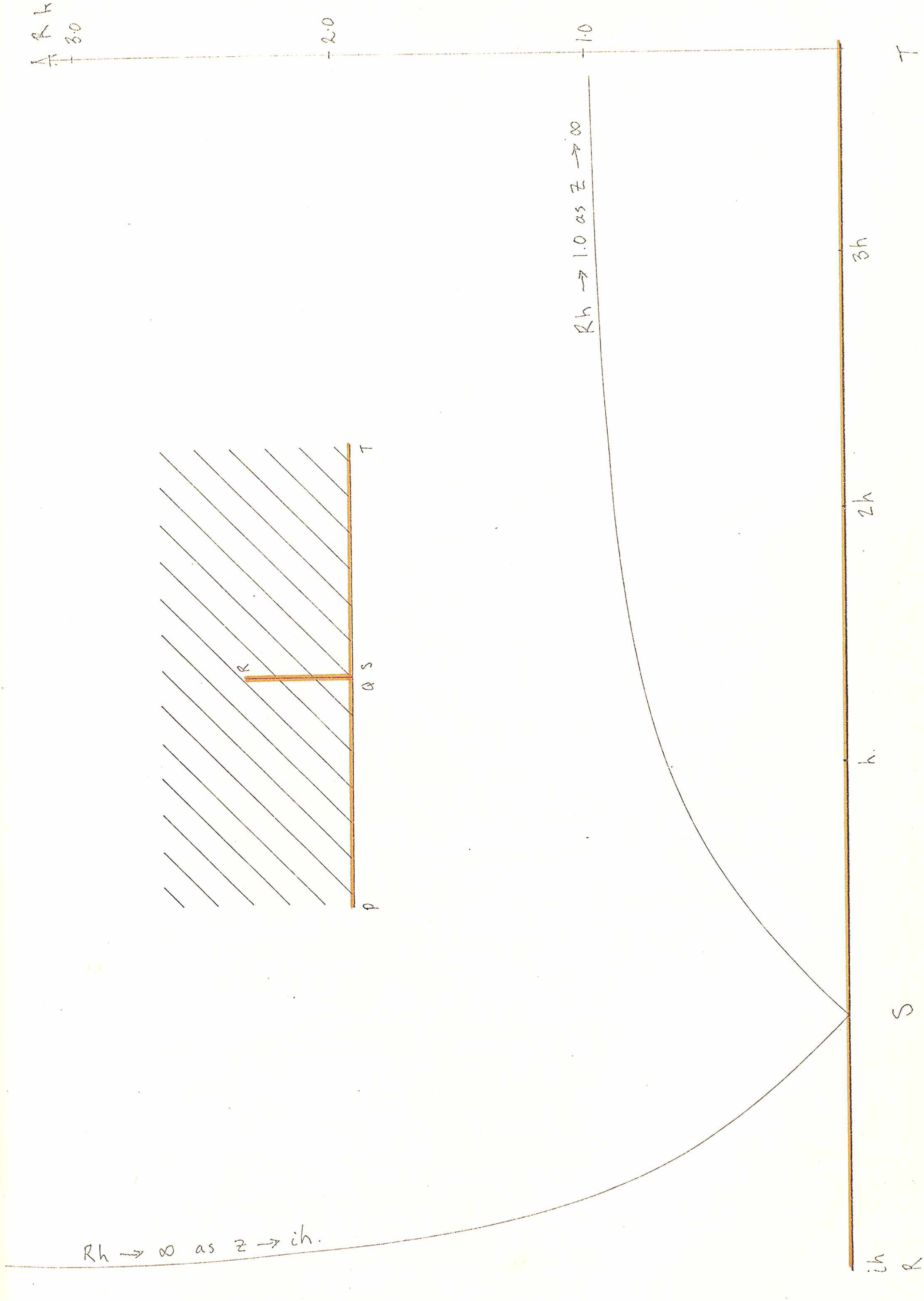
i.e.

$$\frac{dw}{dz} = -\frac{z}{h\sqrt{z^2 + h^2}} \qquad \text{------------} \quad (4.16)$$

Substituting this into equation (4.15) gives

$$R = \frac{1}{h}\left| \frac{z}{\sqrt{z^2 + h^2}} \right| \qquad \text{------------} \quad (4.17)$$

We can now find the field strength at any point along the ground and wall by substituting the value of z required. This was done for various values of z along the ground TS and up the wall SR in figure (4.1a). The calculated values are tabulated in Tables 4.1 and 4.2 and the plot of the results can be seen in Graph 4.2.



Graph 4.2 Variation of Field Strength - Lee's Wall.

z/h	Rh
0.0	0.0
0.10	0.10
0.25	0.24
0.50	0.45
1.00	0.71
1.50	0.83
2.00	0.89
2.50	0.93
3.00	0.95
∞	1.00

Table 4.1 Variation of R along ground TS

z/h	Rh
0.0	0.0
0.10	0.10
0.20	0.20
0.25	0.26
0.50	0.58
0.60	0.75
0.75	1.14
0.90	2.07
0.95	3.16
1.00	∞

Table 4.2 Variation of R along wall SR

Three points can be noted from the graph. Firstly the field strength on the ground tends towards a value of $\frac{1}{h}$ as the distance from the wall tends to infinity. Secondly, the value of field strength at the internal part of a corner, i.e. at point S, is zero and thirdly, the field strength at the external part of a corner, i.e. at R, is infinite.

The charge density σ is given as

$$\sigma = \frac{K}{4\pi} \left| \frac{dw}{dz} \right| \quad \text{------------} \quad (4.18)$$

where K is the dielectric constant of the medium. Hence for the Lee's Wall problem

$$\sigma = \frac{K}{4\pi h} \left| \frac{z}{\sqrt{z^2 + h^2}} \right| \quad \text{---------} \quad (4.19)$$

The variation in charge density is similar to that of the field strength R. The charge Q on any length of the ground or wall is given as

$$Q = \frac{K}{4\pi} \cdot \left| u_2 - u_1 \right| \qquad \text{------------} \quad (4.20)$$

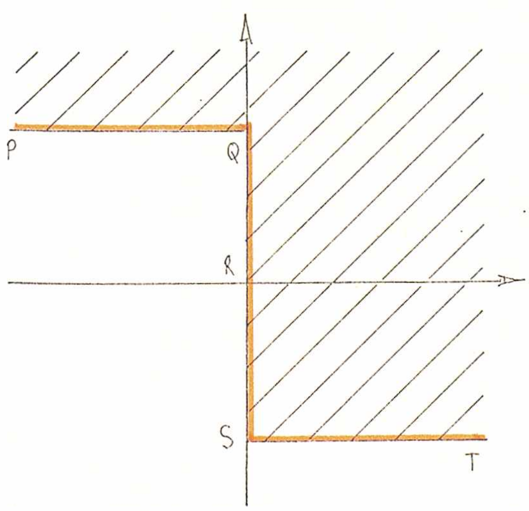
hence if we want the total charge on the wall we have to refer to figure (4.1c) where the wall is represented on the w-plane by the section QRS. The limits u_2 and u_1 are given as Q and S respectively and thus

$$Q = \frac{K}{4\pi} \cdot \left| 1 - (-1) \right|$$

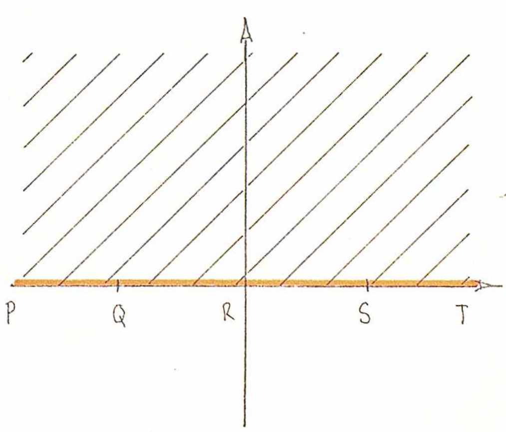
Therefore $Q = \dfrac{K}{2\pi}$

It would appear from this that the total charge on the wall was independent of the height of the wall or the potential but only on the dielectric constant of the medium.

Another problem investigated by Lees was the retaining wall configuration shown in figure (4.2a). Here we have two horizontal plane surfaces PQ and ST separated by a long vertical retaining wall QS. The z-plane



(a) z-plane



(b) t-plane

Figure 4.2 Transformation of retaining wall.

was chosen as shown in figure (4.2a) with the origin half way up the wall. Thus we require to transform the perimeter PQRST in the z-plane on to the real axis of the t-plane. The mapping table becomes:

Point	Q	$t_1 = -1$	$\alpha_1 = \frac{3\pi}{2}$
	S	$t_2 = 1$	$\alpha_2 = \frac{\pi}{2}$

from the Schwarz-Christoffel Transformation we get

$$\frac{dz}{dt} = A(t+1)^{-\frac{1}{2}}(t-1)^{-\frac{1}{2}}$$

$$= A\sqrt{\frac{t+1}{t-1}}$$

Hence $$z = A\int\sqrt{\frac{t+1}{t-1}}\,.\,dt$$

If we let $t = \cosh\theta$ and substitute we get

$$z = A\int\sqrt{\frac{\cosh\theta + 1}{\cosh\theta - 1}}\,.\sinh\theta\,.\,d\theta$$

Multiplying top and bottom by $\sqrt{\cosh\theta + 1}$ gives

$$z = A\int\frac{\cosh\theta + 1}{\sqrt{\cosh^2\theta - 1}}\sinh\theta\ d\theta$$

$$= A\int(\cosh\theta + 1)\cdot d\theta$$

Thus $$z = A(\sinh\theta + \theta) + B$$

Returning to our $z = f(t)$ relationship we get from $t = \cosh\theta$

$$z = A\left(\sqrt{t^2 - 1} + \cosh^{-1}t\right) + B$$

To evaluate the two constants A and B we have to substitute boundary conditions.

1. At point S, $z = -ih$ and $t = 1$

$$-ih = A\,(0 + \cosh^{-1} 1) + B$$

Hence $B = -ih$

2. At point Q, $z = ih$ and $t = -1$

$$ih = A\,(0 + \cosh^{-1} -1) - ih$$

$$2ih = A\,i\pi$$

Therefore $A = \dfrac{2h}{\pi}$

The geometrical transformation $z = f(t)$ is thus

$$z = \frac{2h}{\pi}\left(\sqrt{t^2 - 1} + \cosh^{-1} t\right) - ih \qquad \text{------------} \quad (4.20a)$$

The electrical transformation in this problem is identical with that of the last one and therefore it will be omitted in this analysis. The field strength R is given as

$$R = \left|\frac{dw}{dz}\right|$$

$$= \left|\frac{dw}{dt} \cdot \frac{dt}{dz}\right|$$

Taking $\dfrac{dw}{dt} = -1$ from the last example we get

$$R = \left| -1 \cdot \frac{\pi}{2h} \cdot \sqrt{\frac{t-1}{t+1}} \right|$$

$$= \frac{\pi}{2h} \left| \sqrt{\frac{t-1}{t+1}} \right|$$

As a test we know that the field strength at the external part of a corner is infinite and at the internal part is zero. Therefore R should be infinite at Q and zero at S.

1. At Q, $t = -1$

$$R = \frac{\pi}{2h} \left| \sqrt{\frac{-2}{0}} \right|$$

$$= \infty$$

2. At S, $t = 1$

$$R = \frac{\pi}{2h} \left| \sqrt{\frac{0}{2}} \right|$$

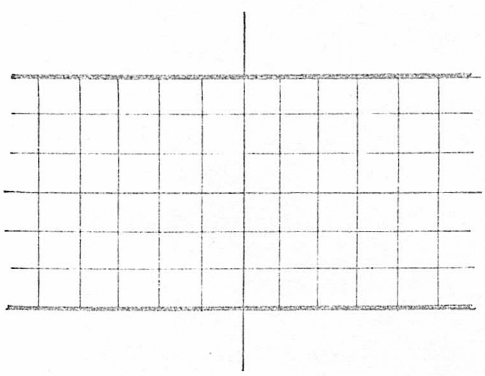
$$= 0$$

Hence the equation holds at these two points.

There are a number of other combinations of horizontal and vertical walls that can be investigated by this process, the main difficulty being the integration of the Schwarz-Christoffel equation $\frac{dz}{dt}$.

4.2 Parallel Plate Capacitor

This is the second of the two examples which illustrates a different set of boundary conditions. In this case we have two identical parallel plates held at different potentials between which there is an electric field. If this is a uniform field then the lines of equipotential and lines of force between the plates form a rectangular grid as shown in figure (4.3a).



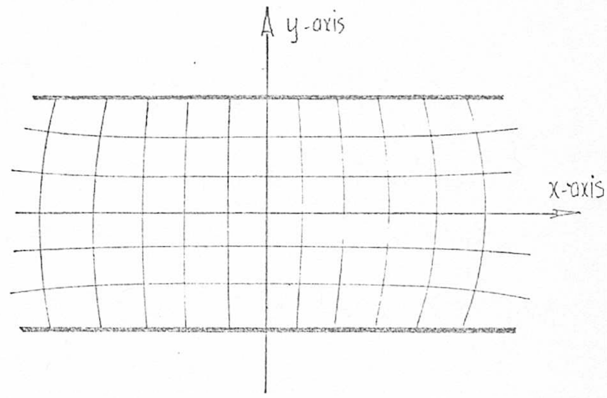


Figure 4.3a Ideal Capacitor

Figure 4.3b Actual Capacitor

In fact the field is rather uneven due to the end effects as illustrated in figure (4.3b). The extent to which the field is distorted is dependant on its distance from the end and, as we shall see later, the distance between the plates. We will consider only the field between a semi-infinite plate and an infinite one since the capacitor is symmetrical about the x and y axis in the plane of the paper.

The z-plane is chosen as shown in figure (4.4a). The points A, B, C and C' are at infinity and the polygon is effectively a triangle with corners at D, C and A. Since C' and A meet at infinity the angle at A is zero. C and B also meet at infinity although they travel by opposite routes. The angle at C is therefore π. The angle at D is 2π.

We require to transform the boundary of this triangle onto the real axis of another complex plane which we will call the t-plane. The Schwarz-Christoffel transformation becomes

$$\frac{dz}{dt} = A(t - t_1)^{\frac{\alpha_1}{\pi}-1} (t - t_2)^{\frac{\alpha_2}{\pi}-1} (t-t_3)^{\frac{\alpha_3}{\pi}-1} \qquad \text{------------- (4.21)}$$

The t-plane is drawn so that the point t = 0 corresponds to the point z = A in the z-plane. For simplicity we let D correspond to t = -1. The t-plane can be seen in Figure (4.4b) where the upper half of the z-plane has been transformed to the upper half of the t-plane.

The mapping table becomes

Point D	$t_1 = -1$	$\alpha_1 = 2\pi$
B	$t_2 = \infty$	$\alpha_2 = \pi$
A	$t_3 = 0$	$\alpha_3 = 0$

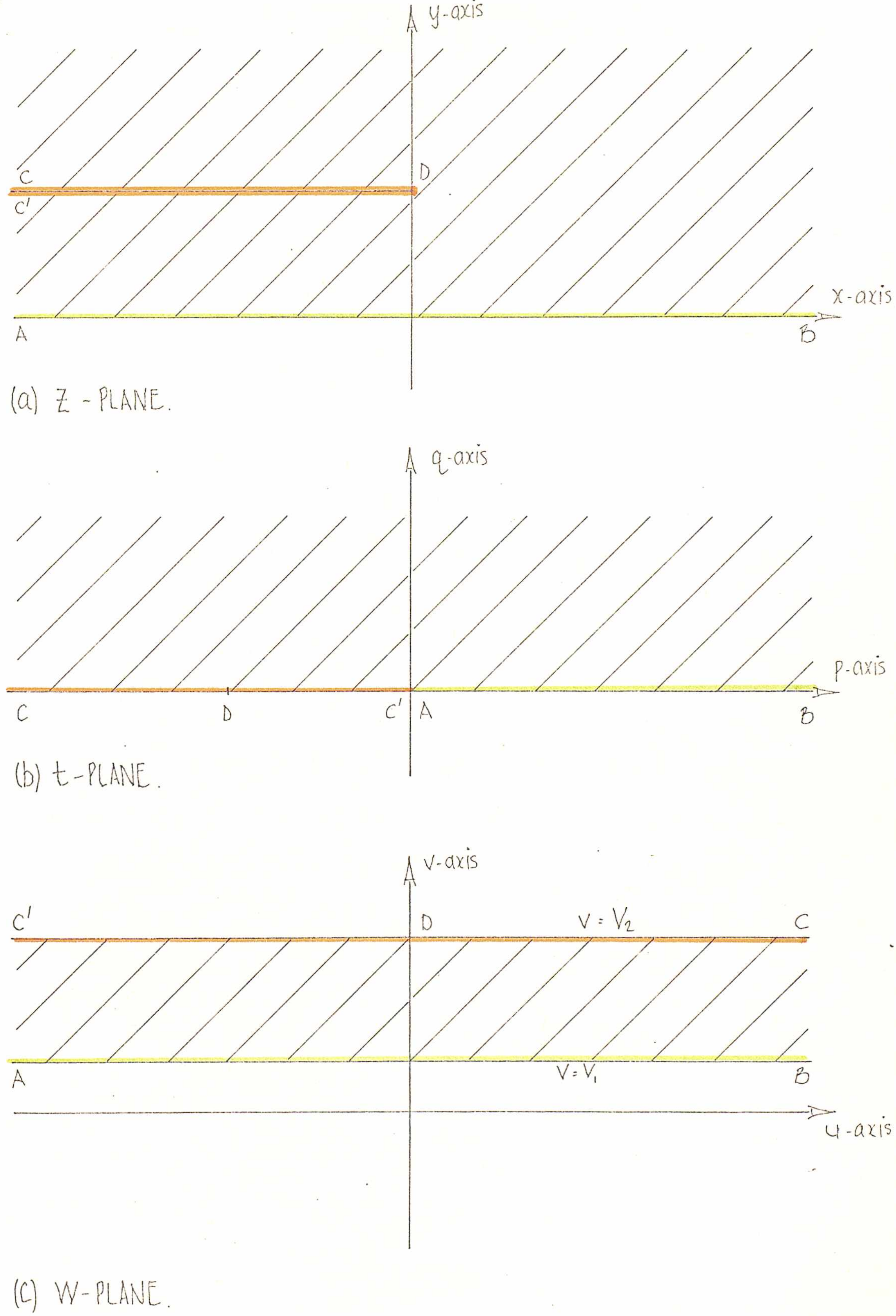


Figure 4.3 Transformation of Capacitor.

Therefore the Schwarz-Christoffel transformation becomes

$$\frac{dz}{dt} = A(t+1)^{\frac{2\pi}{\pi}-1} \quad (t-\infty)^{\frac{\pi}{\pi}-1} \quad (t-0)^{\frac{0}{\pi}-1}$$

$$= A\,\frac{(t+1)}{t}$$

Hence $z = A \int \frac{t+1}{t}\, dt + B$

$$= A\left[t + \ln t\right] + B \qquad \text{------------} \quad (4.22)$$

where A and B are complex constants. To find the values of these constants we must substitute boundary conditions. These are found as follows.

1. At the corner A, z changes in amplitude by ih but not in argument, while t changes in argument by π but not in amplitude. Hence the change in amplitude in the z-plane must equal the change in argument in the t-plane.

ie $$\int_{-\infty+0}^{-\infty+ih} dz = A \int_{\theta=0}^{\theta=\pi} \frac{t+1}{t}\,.\, dt$$

Let $t = |t|\ \exp(i\theta)$, $dt = i\,|t|\,.\exp(i\theta)d\theta$

$$\left[z\right]_{-\infty+0}^{-\infty+ih} = A\int_{\theta=0}^{\theta=\pi} \frac{|t|\ \exp(i\theta)+1}{|t|\ \exp(i\theta)}\,.\,i\,|t|\exp(i\theta)d\theta$$

$$-\infty + ih + \infty - 0 = iA\int_{\theta=0}^{\theta=\pi}\left[|t|\ \exp(i\theta) + 1\right]\ d\theta$$

$$ih = iA\left[\frac{|t|\ \exp(i\theta)}{i}\right]_0^{\pi} + \left[iA\ \theta\right]_0^{\pi}$$

But by definition $|t| = 0$ at the corner A.

Therefore $\quad ih = iA\pi$

and $\quad A = \frac{h}{\pi} \qquad \text{------------} \quad (4.23)$

2. At corner D, $z = ih$ and $t = -1$

$$ih = \frac{h}{\pi}\left[-1 + \ln - 1\right] + B$$

$$= \frac{h}{\pi}\left[-1 + i\pi\right] + B$$

Thus $$B = ih + \frac{h}{\pi} - ih$$

$$B = \frac{h}{\pi} \quad \text{------------} \quad (4.24)$$

We can now substitute the values of A and B into the transformation equation

$$z = \frac{h}{\pi}\left[t + \ln t + 1\right] \quad \text{--------} \quad (4.25)$$

This equation therefore, transforms the semi-infinite plane and the infinite plane in the z-plane onto the real axis of the t-plane. The semi-infinite plane, which is effectively one of the plates of the capacitor, has been transformed onto the negative half of the real axis while the infinite plane, which is the central equipotential of the capacitor system, has been transformed onto the positive half.

We must now introduce a third complex plane to set up the correct electrical transformation. This is necessary since the conductor system, which in the z-plane is at two potentials, occupies the same straight line in the t-plane. We require the negative and positive parts of the real axis of the t-plane to be transformed to different potentials on the electrical w-plane. The effect of this can be seen in figure (4.4c) which shows the two halves of the real axis of the t-plane transformed to two separate infinite lines on the w-plane. The two potentials are V_1 and V_2.

To find the correct transformation equation we again employ the Schwarz-Christoffel relationship this time transforming the two plates on the w-plane onto the real axis of the t-plane.

This time there are only two corners and hence we require to select only two points in the t-plane. The mapping table becomes:

Point A	$t_1 = 0$	$\alpha_1 = 0$
B	$t_2 = \infty$	$\alpha_2 = 0$

Therefore the Schwarz-Christoffel transformation becomes

$$\frac{dw}{dt} = K(t - 0)^{\frac{0}{\pi} - 1} \quad (t - \infty)^{\frac{0}{\pi} - 1}$$

and since any factor with infinity may be ignored

$$\frac{dw}{dt} = K(t)^{-1}$$

or $\quad w = K \ln t + L$ ————————— (4.26)

Again we must find the complex constants K and L by substituting boundary conditions.

1. At the corner A, w changes in amplitude by $(V_2 - V_1)$ but not in argument, while t changes in argument by π but not in amplitude.

Therefore $$\int_{-\infty + iV_1}^{-\infty + iV_2} dw = K \int_{\theta=0}^{\theta=\pi} \frac{dt}{t}$$

Let $t = |t| \exp(i\theta)$, $dt = i \ |t| \ \exp(i\theta) d\theta$

Therefore $$\left[w \right]_{-\infty + iV_1}^{-\infty + iV_2} = K \int_0^{\pi} i \ d\theta$$

$$iV_2 - iV_1 = iK \left[\theta \right]_0^{\pi}$$

$$V_2 - V_1 = K\pi \quad .$$

and/

and $$K = \frac{V_2 - V_1}{\pi} \qquad \text{------------} \quad (4.27)$$

2. At point D, $t = -1$, $w = iV_2$

Therefore $$iV_2 = \frac{V_2 - V_1}{\pi} \ln -1 + L$$

$$iV_2 = \frac{V_2 - V_1}{\pi} \cdot i\pi + L$$

$$L = iV_1 \qquad \text{------------} \quad (4.28)$$

We can now substitute the values of K and L into the transformation equation for w.

Hence $$w = \frac{V_2 - V_1}{\pi} \ln t + iV_1 \qquad \text{------------} \quad (4.29)$$

We have now established the transformation equations between the z and t planes and between the w and t planes. From equations (4.25) and 4.29) we can now eliminate t to find the direct relationship between the z and w planes.

From (4.29) $$w = \frac{V_2 - V_1}{\pi} \ln t + iV_1$$

$$\ln t = \frac{\pi(w - iV_1)}{V_2 - V_1}$$

$$t = \exp\left[\frac{\pi(w - .iV_1)}{V_2 - V_1}\right]$$

Hence substituting into (4.25) we get

$$z = \frac{h}{\pi}\left[\exp\left[\frac{\pi(w - iV_1)}{V_2 - V_1}\right] + \frac{\pi(w - iV_1)}{V_2 - V_1} + 1\right]$$

For simplicity let $V_1 = 0$ and $V_2 = 1$.

Therefore $z = \frac{h}{\pi}\left[\exp \pi w + \pi w + 1\right]$ -------------- (4.30)

If we separate the real and imaginary parts of this equation we can get the x and y coordinates in the z-plane corresponding to the u and v coordinates in the w-plane. Thus

$$x + iy = \frac{h}{\pi}\left[\exp \pi(u + iv) + \pi(u + iv) + 1\right]$$
$$= \frac{h}{\pi}\left[\exp \pi u\ (\cos \pi v + i \sin \pi v) + \pi u + i\pi v + 1\right]$$

Therefore

$$x = \frac{h}{\pi}\left[\exp\ (\pi u).\cos \pi v + \pi u + 1\right] \qquad \text{-------------- (4.31)}$$

and

$$y = \frac{h}{\pi}\left[\exp\ (\pi u).\sin \pi v + \pi v\right] \qquad \text{-------------- (4.32)}$$

By substituting different values of u for a given value of v we can plot the equipotentials in the z-plane. By using the same values of u but varying the value of v, we can also obtain the lines of force.

This was in fact carried out for the capacitor system shown in figure (4.4a). A simple computer program was written to calculate the coordinates x and y for different values of u and v. These were plotted and the results can be seen in Graph 4.3.

Having found the transformation equation we can now use it to calculate other properties of the electrostatic field. The field strength R is given by

$$R = \left| \frac{dw}{dz} \right|$$

This can be written as

$$R = \left| \frac{dw}{dt} \cdot \frac{dt}{dz} \right|$$

But $\frac{dw}{dt} = \frac{(V_2 - V_1)}{\pi} \cdot \frac{1}{t}$

and $\frac{dt}{dz} = \frac{\pi}{h} \cdot \frac{t}{(t + 1)}$

$$\text{Therefore} \quad R = \left| \frac{(V_2 - V_1)}{\pi} \cdot \frac{1}{t} \cdot \frac{\pi}{h} \cdot \frac{t}{(t + 1)} \right|$$

$$= \frac{V_2 - V_1}{h} \left| \frac{1}{t + 1} \right|$$

But we let $V_2 = 1$ and $V_1 = 0$

$$\text{hence} \quad R = \frac{1}{h} \left| \frac{1}{t + 1} \right| \qquad (4.33)$$

We can therefore find the field strength at any point in the vicinity of the capacitor system. As a quick check of our relation we know that the field strength should be infinite at the external part of a corner, and hence at point D in figure (4.4a) R should be infinite.

At point D, from figure (4.4b), $t = -1$

$$\text{Therefore} \quad R = \frac{1}{h} \left| \frac{1}{-1 + 1} \right|$$

$$= \infty$$

Thus the equation for R holds at this point. By selecting suitable values of t we can plot the field strength along the inside and outside surfaces of the capacitor and also along the zero equipotential. This will indicate how far in from the end of the capacitor one has to go before the variation in field strength is negligible.

Tables 4.3, 4.4 and 4.5 show the results obtained for field strength along the three surfaces of the capacitor system. The values of the real part of z corresponding to the chosen values of t were also calculated from equation (4.25).

Graph 4.4 shows a plot of the end of the capacitor. It can be seen that inside the capacitor system the field strength tends everywhere on and between the two plates to a singular value equal to the reciprocal of the distance between the two plates.

As we approach the end of the capacitor the field strength increases to infinity on the inside and outside surfaces of the upper plate. This is due to the effects of a sharp corner at D. Along the zero equipotential the field strength at the end of the capacitor decreases to approximately 80% of the internal value, compared with a sudden drop to zero in the ideal capacitor case.

t	z/h	Rh	t	z/h	Rh
0.00	$-\infty$	1.0	-1.00	0.00	∞
-0.005	-1.37	1.005	-1.10	-0.0016	10.0
-0.01	-1.15	1.01	-1.50	-0.03	2.0
-0.02	-0.93	1.02	-1.70	-0.054	1.43
-0.04	-0.72	1.04	-1.90	-0.0825	1.11
-0.05	-0.65	1.05	-2.00	-0.098	1.0
-0.07	-0.55	1.075	-2.5	-0.186	0.66
-0.09	-0.475	1.10	-3.0	-0.286	0.50
-0.10	-0.445	1.11	-4.0	-0.515	0.33
-0.20	-0.26	1.25	-5.0	-0.76	0.25
-0.50	-0.06	2.0	-10.0	-2.13	0.11
-1.00	0.0	∞	$-\infty$	$-\infty$	0.0

Table 4.3 Variation of R along inside surface of plate

Table 4.4 Variation of R along outside surface of plate

t	z/h	Rh	x	c/c_0
0.0	$-\infty$	1.0	0.0	∞
0.01	-1.14	0.99	0.25	3.01
0.02	-0.92	0.98	0.50	2.236
0.05	-0.62	0.952	1.0	1.77
0.10	-0.38	0.91	2.0	1.475
0.11	-0.35	0.90	3.0	1.355
0.20	-0.13	0.835	5.0	1.243
0.30	+0.035	0.77	10.0	1.143
0.40	0.154	0.715	15.0	1.103
0.50	0.257	0.67	20.0	1.082
1.0	0.635	0.50	30.0	1.059
∞	∞	0.0	∞	1.0

Table 4.5 Variation of R along zero equipotential

Table 4.6 Variation of c/c_0 with length of plate x (see p.66)

What we really want to establish in this analysis is a correct mathematical formula for the capacitance of the parallel plate capacitor, which takes into account the fringing field at the end of the plates. To do this we have to find the total charge on the required section of the semi-infinite plate. Since in our transformations we split the plate into upper and lower parts we will derive from first principles the total charge Q_1 on the lower side of the plate and merely indicate the corresponding value on the upper side.

The total charge Q_1 on the lower side is found by integrating the charge density along a chosen length of plate designated by two values of u on the w-plane.

Thus from equation (2.13)

$$Q_1 = \frac{1}{4} \quad (u_1 - u_2) \qquad \text{-----------------} \quad (4.34)$$

We require to find the corresponding values on the z-plane and this is accomplished via the t-plane. From equation (4.29) with $V_2 = 1$ and $V_1 = 0$ we have

$$w = \frac{1}{\pi} \ \ln t$$

Since q = 0 on the surface of the conductor

$$u = \frac{1}{\pi} \ln p$$

Substituting into equation (4.34) gives

$$Q_1 = \frac{1}{4\pi^2} \ \ln \frac{p_1}{p_2}$$

If we wish to find the total charge from the edge of the plate to some value inside, the numerical values of p become:

$p_1 = -1$, and p_2 = some negative value where $-1 < p_2 < 0$.

Hence $$Q_1 = \frac{1}{4\pi^2} \ \ln \left| \frac{1}{p_2} \right| \qquad \text{-----------} \quad (4.35)$$

where $\left| \frac{1}{p_2} \right|$ signifies the absolute value of $\frac{1}{p_2}$.

The problem now is one of finding p in terms of x. This can be seen from figure 4.5 which shows the length of plate in question in both the t and z planes. At the moment we have the total charge Q_1 on the underside of the plate from D on the t-plane to some arbitrary value of p where $-1 < p < 0$. We have to find the corresponding point x on the z-plane where $-\infty < x < 0$. The transformation equation between the z and t planes was given as

$$z = \frac{h}{\pi} \left[t + \ln t + 1 \right].$$

this can be written as

$$x + iy = \frac{h}{\pi}\left[p + \ln p + 1 \right]$$

or, since p is negative

$$x + iy = \frac{h}{\pi}\left[-p + i\pi - \ln \left| \frac{1}{p} \right| + 1 \right]$$

Hence $x = \frac{h}{\pi}\left[-p - \ln \left| \frac{1}{p} \right| + 1 \right]$ ——————— (4.36)

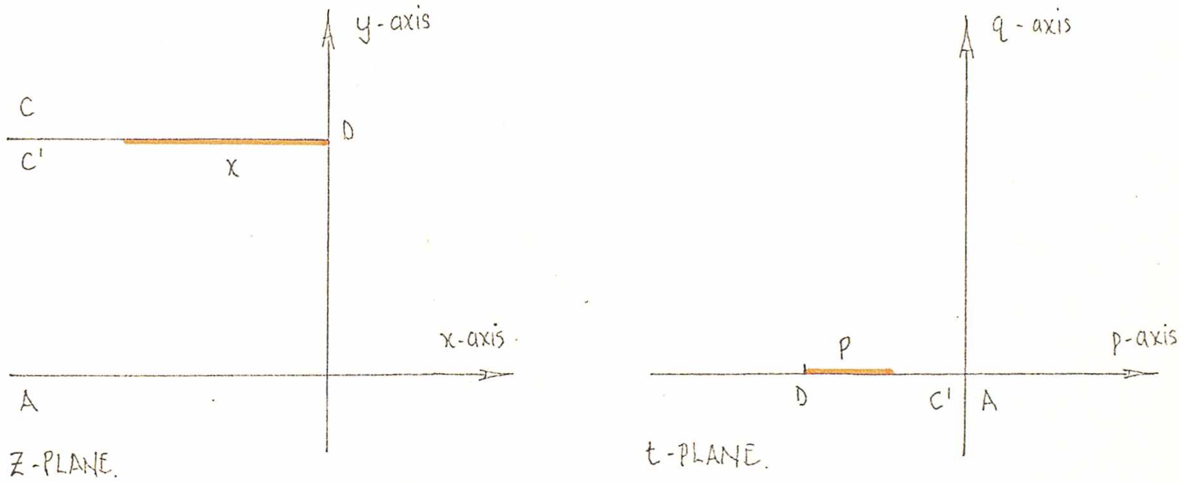


Figure 4.5 Length of plate for which charge is calculated.

We have now found the relation between x and p and require p as a function of x. From the above relation this could prove rather difficult, but if we graph x against p, we find that for values of $|x| > 1$, $|p| < 0.02$ and hence $\ln \left| \frac{1}{p} \right| \gg |p|$.

We can now write

$$x = \frac{h}{\pi}\left[1 - \ln \left| \frac{1}{p} \right| \right]$$

or

$$\ln \left| \frac{1}{p} \right| = 1 - \frac{x\pi}{h}$$

Substituting this into equation (4.35) we get

$$Q_1 = \frac{1}{4\pi^2}\left[1 - \frac{x\pi}{h} \right]$$

or since x will be negative

$$Q_1 = \frac{1}{4\pi^2}\left[1 + \frac{\pi}{h}|x|\right] \quad \text{------------- (4.37)}$$

where $|x|$ signifies the numerical value of x as measured from D in the z-plane.

Thus we have found the total charge Q_1 on the lower side of the plate. The total charge Q_2 on the corresponding length of the upper side is found similarly and is given as

$$Q_2 = \frac{1}{4\pi^2}\ \ln\left[1 + \frac{\pi}{h}|x|\right] \quad \text{------------- (4.38)}$$

Hence the total charge on the plate is simply the sum of Q_1 and Q_2

ie $$Q = \frac{1}{4\pi^2}\left[\ln(1 + \frac{\pi}{h}|x|) + (1 + \frac{\pi}{h}|x|)\right] \text{------ (4.39)}$$

The capacitance of this length of capacitor is found by dividing the total charge Q by the potential difference between the plates.

$$C = \frac{Q}{V}$$

But for our capacitor V = 1, therefore, assuming unit thickness of capacitor

$$C = \frac{1}{4\pi^2}\left[\ln(1 + \frac{\pi}{h}|x|) + (1 + \frac{\pi}{h}|x|)\right] \text{-------- (4.40)}$$

The capacitance of an ideal capacitor of equal length is given as

$$C_0 = \frac{x}{4\pi h} \quad \text{------------- (4.41)}$$

Thus $C = C_0\left[\frac{h}{\pi}\left|\frac{1}{x}\right| \ln(1 + \frac{\pi}{h}|x|) + (\frac{h}{\pi}\left|\frac{1}{x}\right| + 1)\right]$

It can be seen that the variables in the above equation are x, the length of the plate, and h, the distance between the plates. A graph of C/C_0 against x/h will give an indication of the effect of the fringing field on the total capacitance of a given length of capacitor. Obviously in the ideal case C/C_0 should be 1.0. If we let h = 1 and vary x then our graph of C/C_0 should tend to 1.0 as $x \to \infty$. The results are shown in Table 4.6 and Graph 4.5. It will be seen from Graph 4.5 that C/C_0 tends to infinity as x tends to zero. This is due to the fact that $C_0 = 0$ at $x = 0$ from equation (4.41).

In fact the actual capacitance C is given from equation (4.40) as

$$C = \frac{1}{4\pi^2}$$

As a practical application of this theory we will examine briefly the subject of dielectric heating and the concept of "safe distance". Dielectric heating is a method of rapid and uniform heating by an electrical method involving the use of radio frequencies. The high frequency output is fed to the plates of a capacitor between which is placed the material to be heated, which therefore becomes the dielectric. The rapidly alternating electric field causes the dipoles in the dielectric to vibrate and oscillate thereby generating heat in the material. For a given frequency it can be shown that the rate of generation of heat per unit volume at any point in the field is proportional to the square of the field strength at that point, and therefore uniformity of heating is dependent on the uniform distribution of field strength between the plates of the capacitor.

From the analysis of the distribution of field strength in the capacitor it was found that

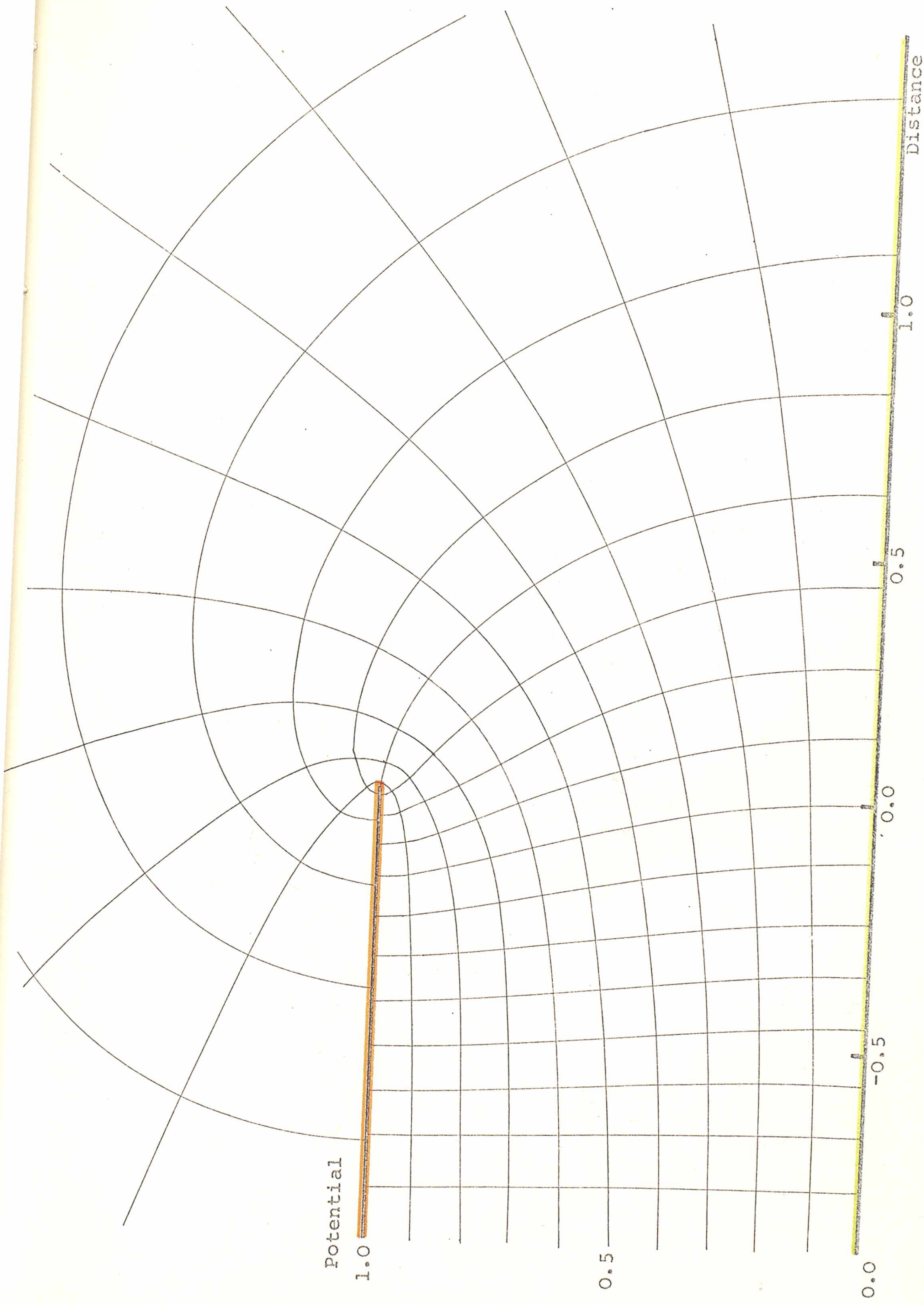
$$R = \frac{1}{h}\left|\frac{1}{t+1}\right|$$

where h is the distance between the plates. From Graph 4.4 we see that R tends everywhere on and between the plates to a singular value equal to $\frac{1}{h}$. As we approach the end of the capacitor, R increases on the inside surface of the upper plate but decreases along the equipotential. If uniformity of heating is to be achieved we must avoid this area of changing field strength.

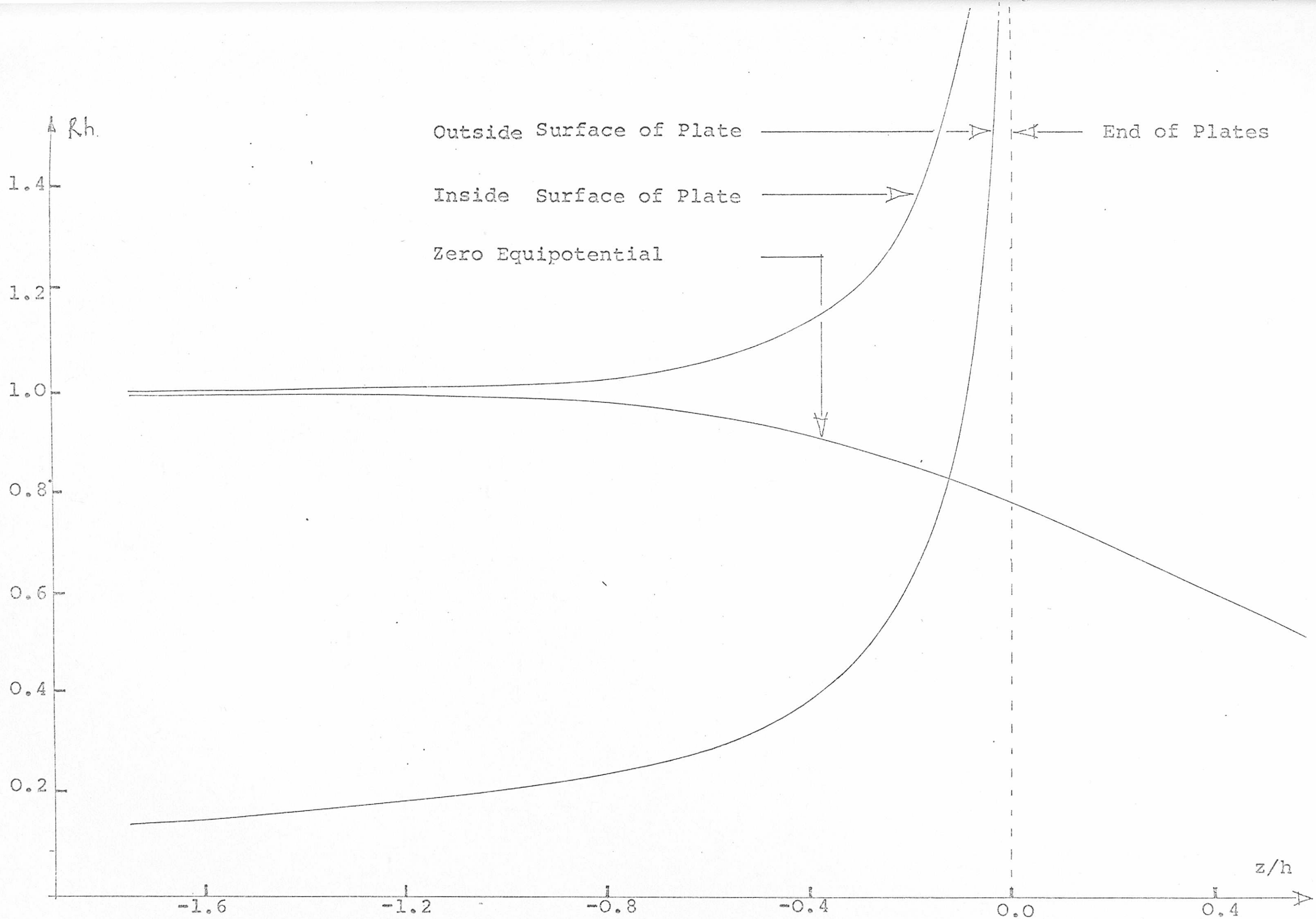
To do this we set an upper limit of acceptable field strength variation, say 1%, and calculate how far in from the end of the capacitor we have to go before R increases or decreases by this amount. Beyond this the variation will be less than 1% and uniformity can be assumed. From Table 4.3 we see that for Rh = 1.01 which is a 1% increase in R the value of z is given as -1.15h. From Table 4.5 we see similarly that a 1% decrease in R along the zero equipotential occurs when z = -1.14h. This shows that for uniform heating the material should be placed a distance of at least 1.15h from the end of the capacitor. Hence 1.15h becomes the safe distance for the capacitor.

This completes the analysis of the Lee's Wall and capacitor problems and the investigation of the use of the Schwarz-Christoffel Transformation in solving two different forms of the electrical transformation.

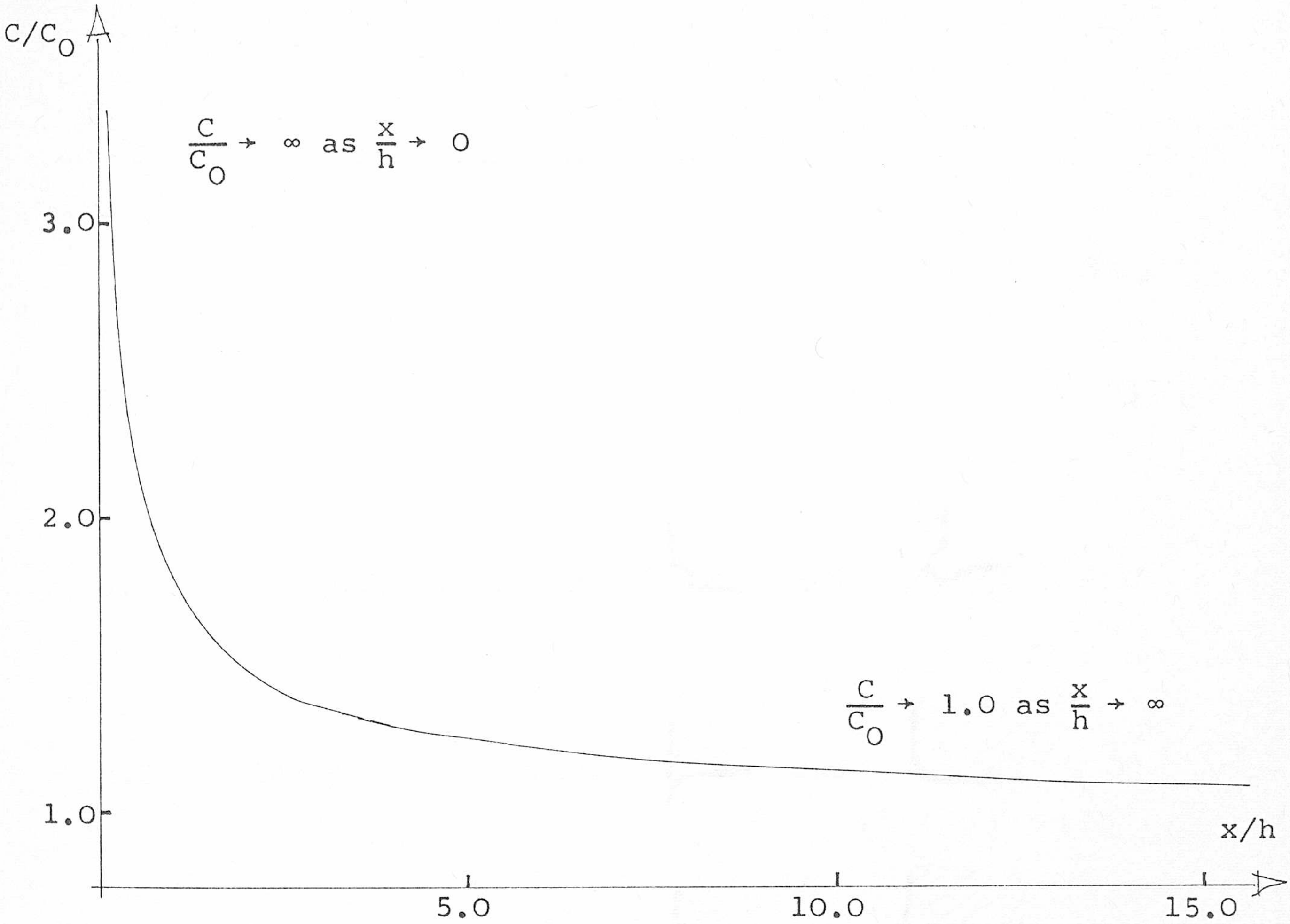
In the next chapter we will investigate several corner configurations exemplifying the range of methods of solution outlined in Chapter 3.



Graph 4.3 Equipotentials and Lines of Force



Graph 4.4 Variation of Field Strength



Graph 4.5 Increase in Capacitance with Length of Plate

CHAPTER 5

VARIATIONS AT A CORNER

Page

VARIATIONS AT A CORNER

This chapter forms the main body of the thesis and involves a study of various right-angled corner shapes. The corner shapes provide, in their analysis, an example of the use of each of the methods of solution listed in Chapter 3. The analysis in this chapter involves finding the transformation equation $z = f(w)$ for each of the corner shapes and using it to establish the field strength around the corner.

Because of the complexity of some of the problems and to avoid unnecessary repetition it was decided to restrict sections 5.1 and 5.2 to establishing the field strength in equation form only while providing a field plot of the area around the corner. Sections 5.3, and 5.4 contain no field plot but a more detailed analysis of the field strength variations including graphs. Section 5.5 contains only the final equations.

5.1 Fields of simple right-angled corner

We will analyse briefly the internal and external fields of the simple corner.

Geometrical Transformation for Internal Field

This transforms the conductor ABC in the z-plane of figure 5.1 onto the real axis of the t-plane with the relevant field becoming the upper half of the t-plane. Using the Schwarz-Christoffel equation we see that with only one angle $\pi/2$ the equation reduces to:

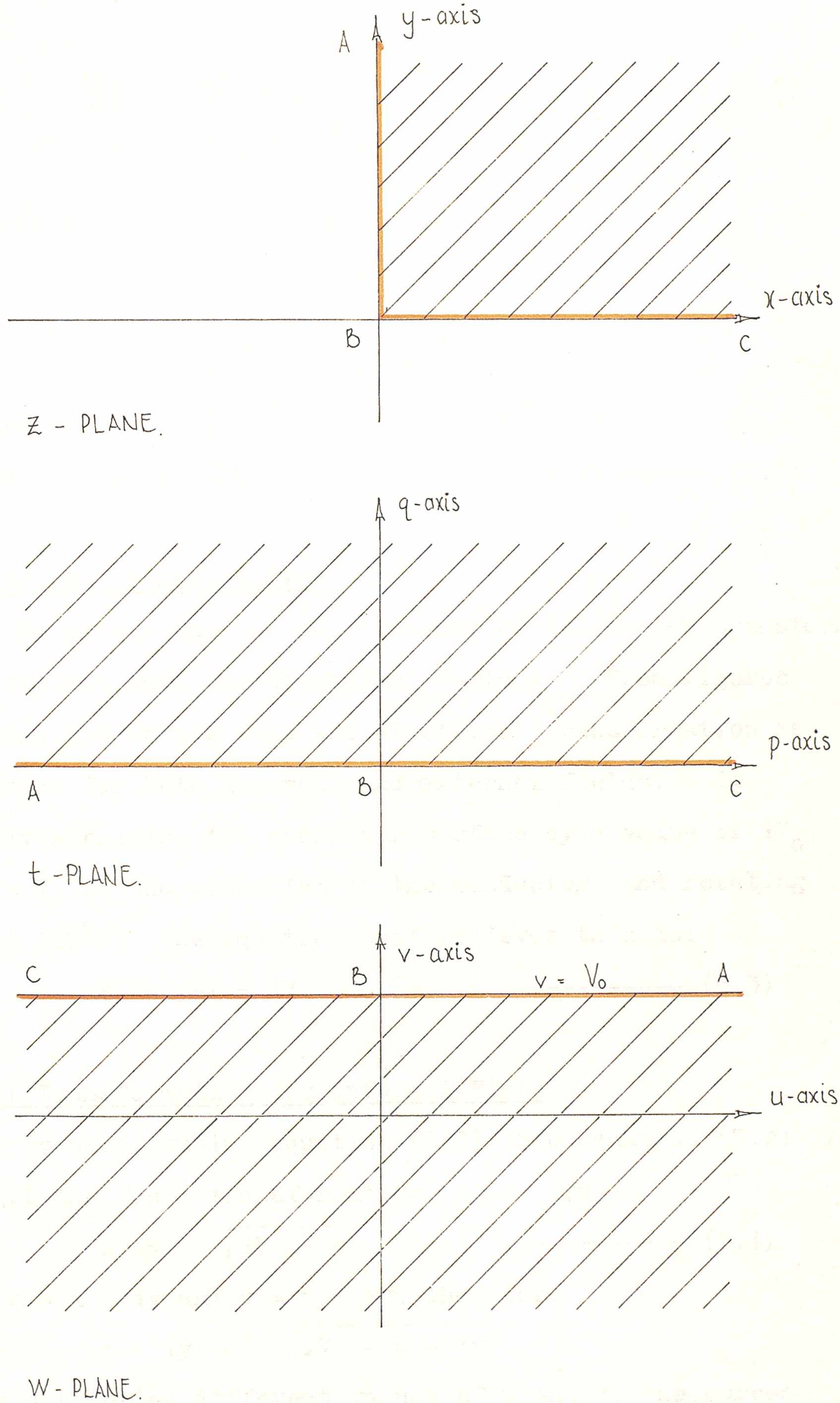


Figure 5.1 Internal Field of Simple Corner.

$$\frac{dz}{dt} = D(t - 0)^{-\frac{1}{2}} \qquad \text{------------- (5.1)}$$

$$z = D \int \frac{1}{\sqrt{t}} \, dt$$

Therefore $z = 2D\sqrt{t} + E$

when $z = 0, \quad t = 0$

Therefore $E = 0$

when $Z = 1, \quad t = 1$

therefore $D = \frac{1}{2}$

hence $Z = \sqrt{t}$ ------------- (5.2)

Electrical Transformation

To obtain the correct boundary conditions we transform the t-plane onto the electrical w-plane. From figures 5.1 and 5.2 we see that the electrical transformation is the same for both internal and external fields. It involves raising the conductor surface by a value of iV_o, where V_o is the potential of the conductor, and rotating about 180^o. The equation that achieves this is:

$$t = -w + iV_o \qquad \text{----------- (5.3)}$$

Final Transformation for Internal Field

By substituting equation (5.3) into equation (5.2) we get the final transformation $z = f(w)$

$$z = \sqrt{iV_o - w} \qquad \text{----------- (5.4)}$$

But $z = x + iy$ and $w = u + iv$, therefore

$$x + iy = \sqrt{iV_o - u - iv}$$

By substituting different values of u and v, the corresponding values of x and y can be found and a field plot of the corner made. This can be seen in Graph 5.1.

Geometrical Transformation for External Field

Again only one corner involved, this time with angle $3\pi/2$ and hence the Schwarz-Christoffel equation becomes:

$$\frac{dz}{dt} = D(t - 0)^{\frac{1}{2}} \qquad \text{------------ (5.5)}$$

$$z = D\int \sqrt{t} \quad dt$$

Therefore $z = \frac{2}{3} D\, t^{\frac{3}{2}} + E$

when $z = 0, \quad t = 0$

therefore $E = 0$

when $z = i, \quad t = 1$

therefore $D = \frac{3}{2} i$

hence $z = i \sqrt{t^3}$ ------------ (5.6)

Final Transformation for External Field

By substituting equation (5.3) into equation (5.6) we get the final transformation $z = f(w)$

$$z = i \sqrt{(iV_o - w)^3} \qquad \text{------------ (5.7)}$$

Hence $x + iy = i \sqrt{(iV_o - u - iv)^3}$

A field plot was made using different values of u and v. This can be seen in Graph 5.2.

Field Strength for Internal Field

From chapter 2 we recall that field strength R was given by

$$R = \left| \frac{dw}{dz} \right|$$

From equation (5.4) we get

$$w = iV_o - z^2$$

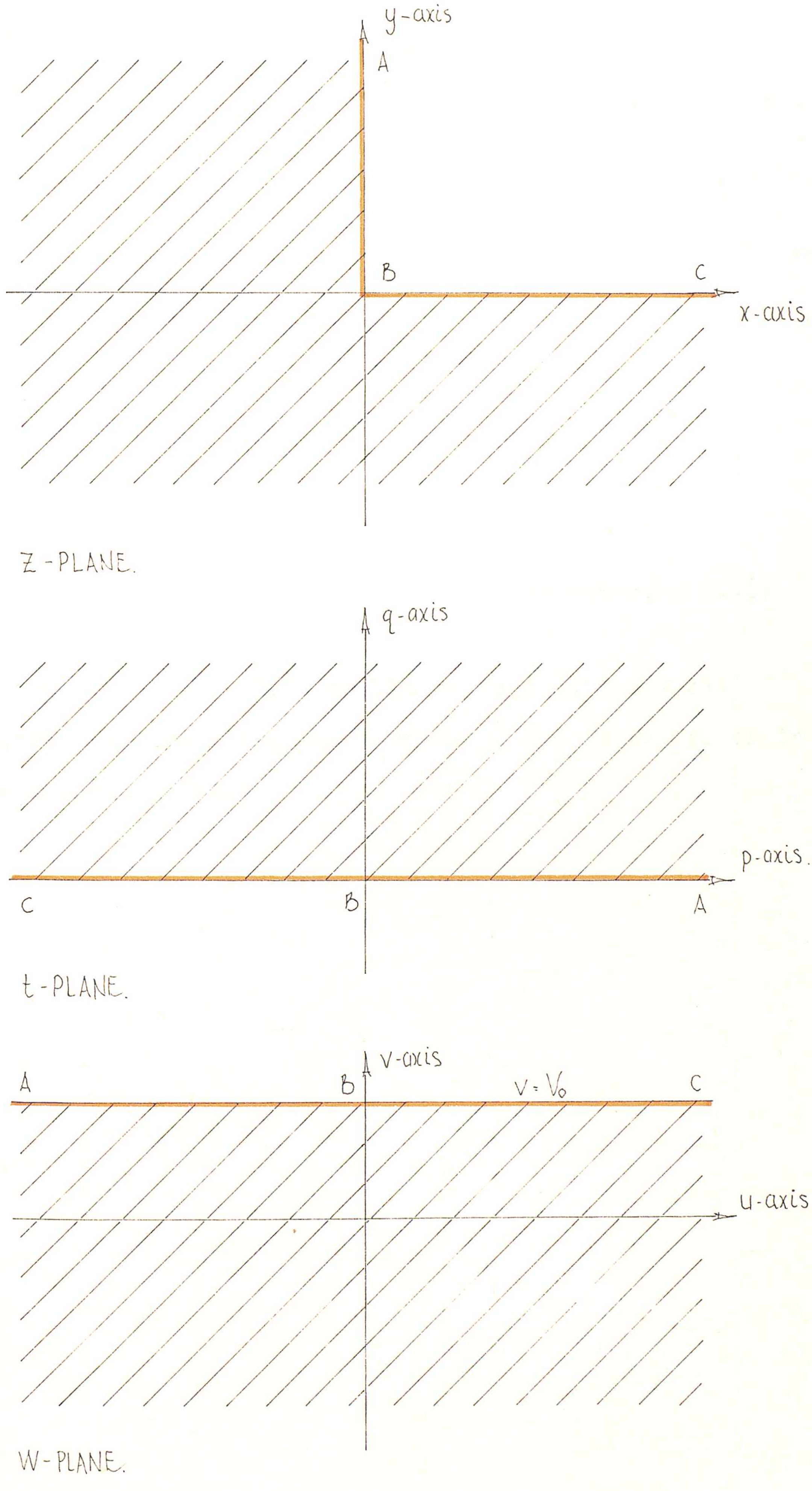


Figure 5.2 External Field of Simple Corner.

and $\frac{dw}{dz} = -2z$

Hence $R_I = |2z|$ ——————— (5.8)

<u>Field Strength for External Field</u>

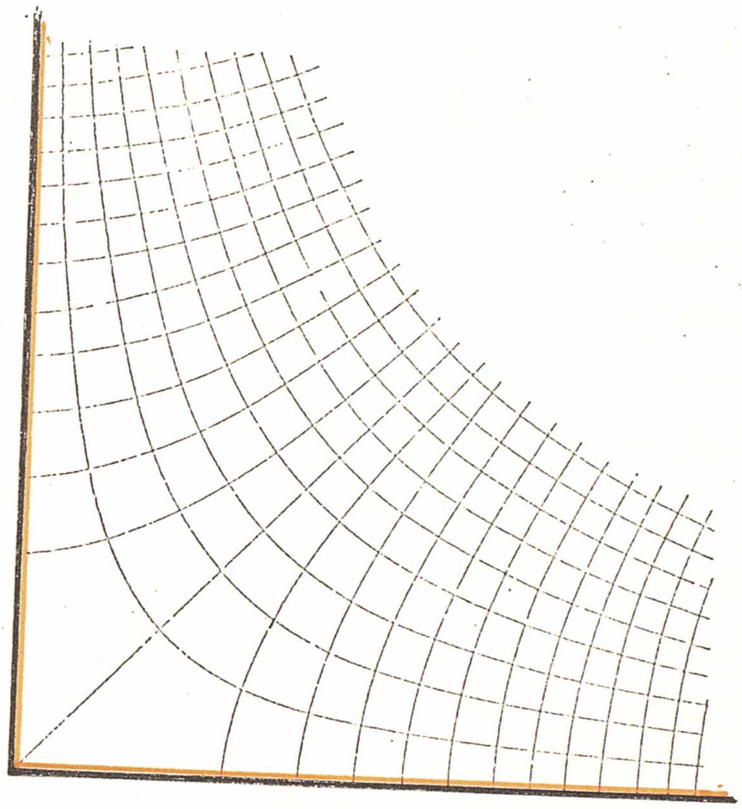
From equation (5.7) we get

$$w = iV_0 - (-1)^{\frac{1}{3}} z^{\frac{2}{3}}$$

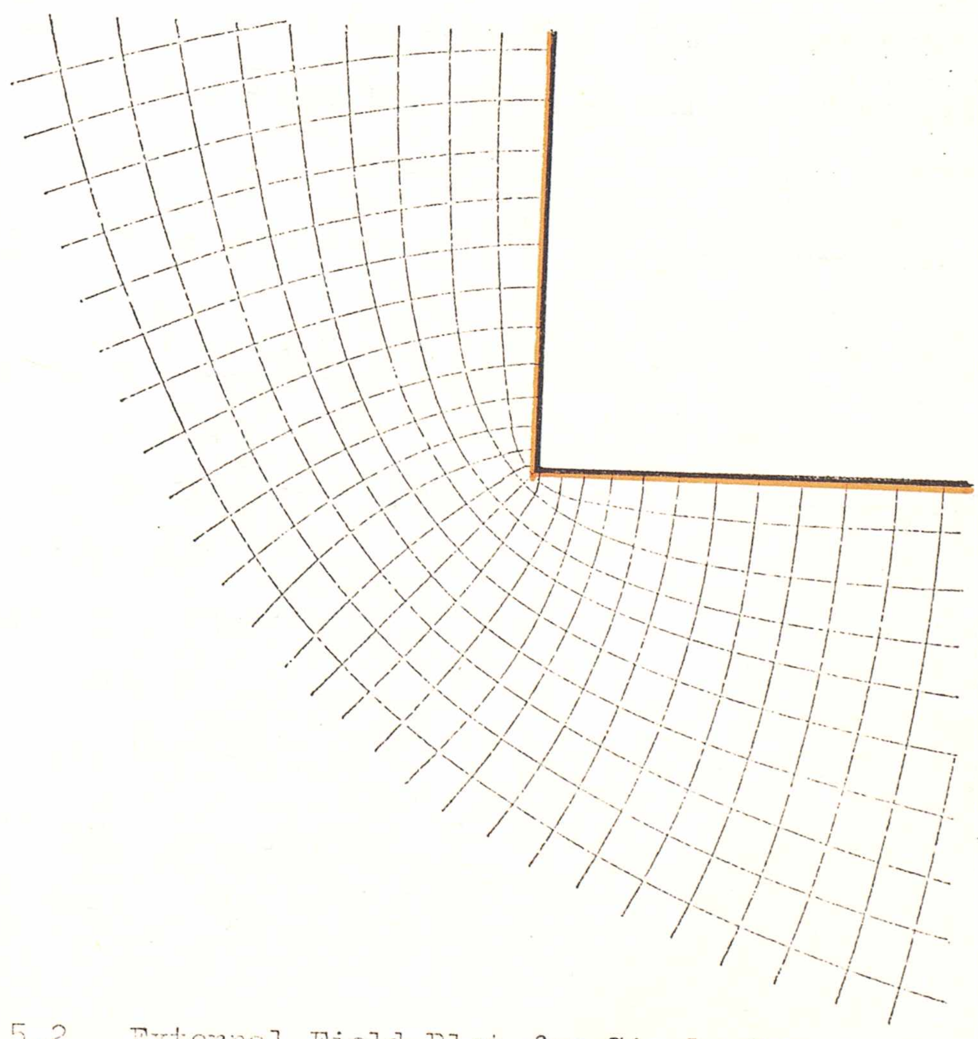
and $\frac{dw}{dz} = -\frac{2}{3} (-1)^{\frac{1}{3}} z^{-\frac{1}{3}}$

Hence $R_E = \frac{2}{3} \left| z^{-\frac{1}{3}} \right|$ ——————— (5.9)

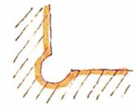
Equations (5.8) and (5.9) enable us to find the field strength anywhere along the surface of the conductor or in the surrounding field.



Graph 5.1 Internal Field Plot for Simple Corner



Graph 5.2 External Field Plot for Simple Corner.

5.2 Analysis of corners  and 

These corners both require the Richmond Method which involves combining the Schwarz-Christoffel transformation with a known transformation equation. The known transformation z = exp.t transforms concentric circles in the z-plane into lines parallel to the q-axis in the t-plane, while the radii become lines parallel to the p-axis. Thus the conductor and field in the z-plane is transformed into the t-plane by the equation:

$$z = \exp . t \qquad \text{------------------} \quad (5.10)$$

Geometrical Transformation for 

Using the Schwarz-Christoffel equation we transform the conductor outline ABCD in the t-plane of figure 5.3 on to the real axis of the c-plane. The corners at B and C with angles $\frac{\pi}{2}$ are located at -1 and +1 respectively on the c-plane. This gives:

$$\frac{dt}{dc} = E\,(c + 1)^{-\frac{1}{2}}\,(c - 1)^{-\frac{1}{2}} \qquad \text{------------------} \quad (5.11)$$

and
$$t = E \int \frac{dc}{\sqrt{c^2 - 1}}$$

$$= E \ln \left[c + \sqrt{c^2 - 1} \right] + F \qquad \text{------------} \quad (5.12)$$

when c = 1, t = 0

therefore $0 = E \ln 1 + F$

and $F = 0$

when c = -1, $t = \frac{i\pi}{2}$

$$\frac{i\pi}{2} = E \ln(-1)$$

and $E = \frac{1}{2}$

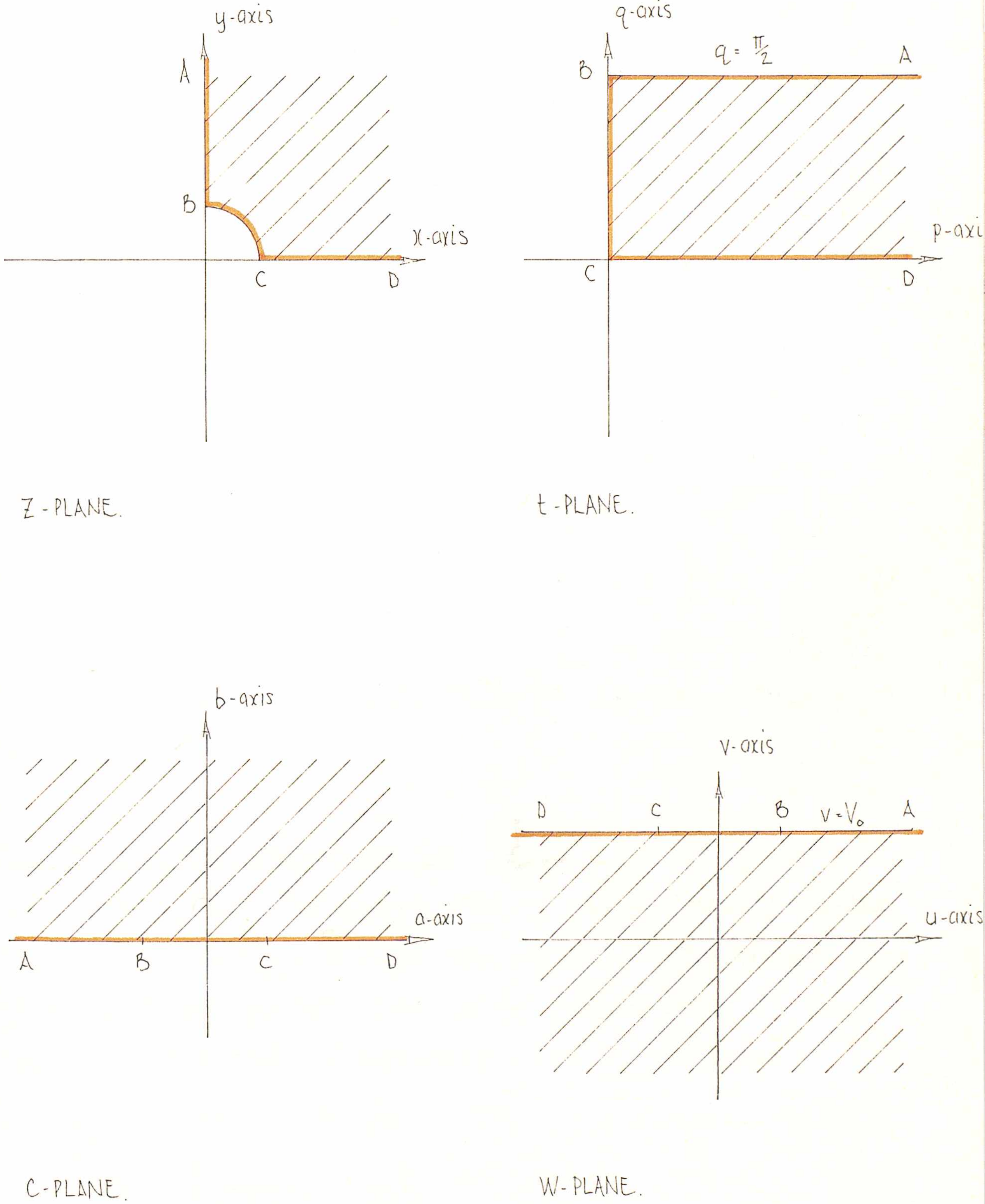
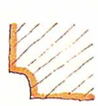


Figure 5.3 Field of 

substituting into equation (5.12) gives

$$t = \frac{1}{2} \ln \left[c + \sqrt{c^2 - 1} \right] \quad \text{------------} \quad (5.13)$$

<u>Electrical Transformation</u>

From equation (5.3) we get

$$c = iV_o - w \quad \text{------------} \quad (5.14)$$

<u>Final Transformation</u>

By substituting equation (5.14) into equation (5.13) we get:

$$t = \frac{1}{2} \ln \left[i V_o - w + \sqrt{(i V_o - w)^2 - 1} \right]$$

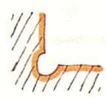
With our known transformation equation (5.10) we have

$$z = \exp \frac{1}{2} \ln \left[i V_o - w + \sqrt{(i V_o - w)^2 - 1} \right]$$

$$\text{Therefore } z = \left[i V_o - w + \sqrt{(i V_o - w)^2 - 1} \right]^{\frac{1}{2}}$$

$$\text{------------} \quad (5.15)$$

Graph 5.3 shows the internal field plot obtained from this equation.

<u>Geometrical Transformation for</u> 

We transform the figure ABCD in the t-plane of figure (5.4) on to the real axis of the c-plane. The corners at B and C are located at +1 and -1 respectively on the c-plane. The Schwarz-Christoffel equation becomes

$$\frac{dt}{dc} = E (c + 1)^{-\frac{1}{2}} (c - 1)^{-\frac{1}{2}} \quad \text{------------} \quad (5.16)$$

$$t = E \int \frac{dc}{\sqrt{c^2 - 1}}$$

$$= E \ln \left[c + \sqrt{c^2 - 1} \right] + F \quad \text{------------} \quad (5.17)$$

when $c = 1$, $t = \frac{i\pi}{2}$

$$\frac{i\pi}{2} = E \ln 1 + F$$

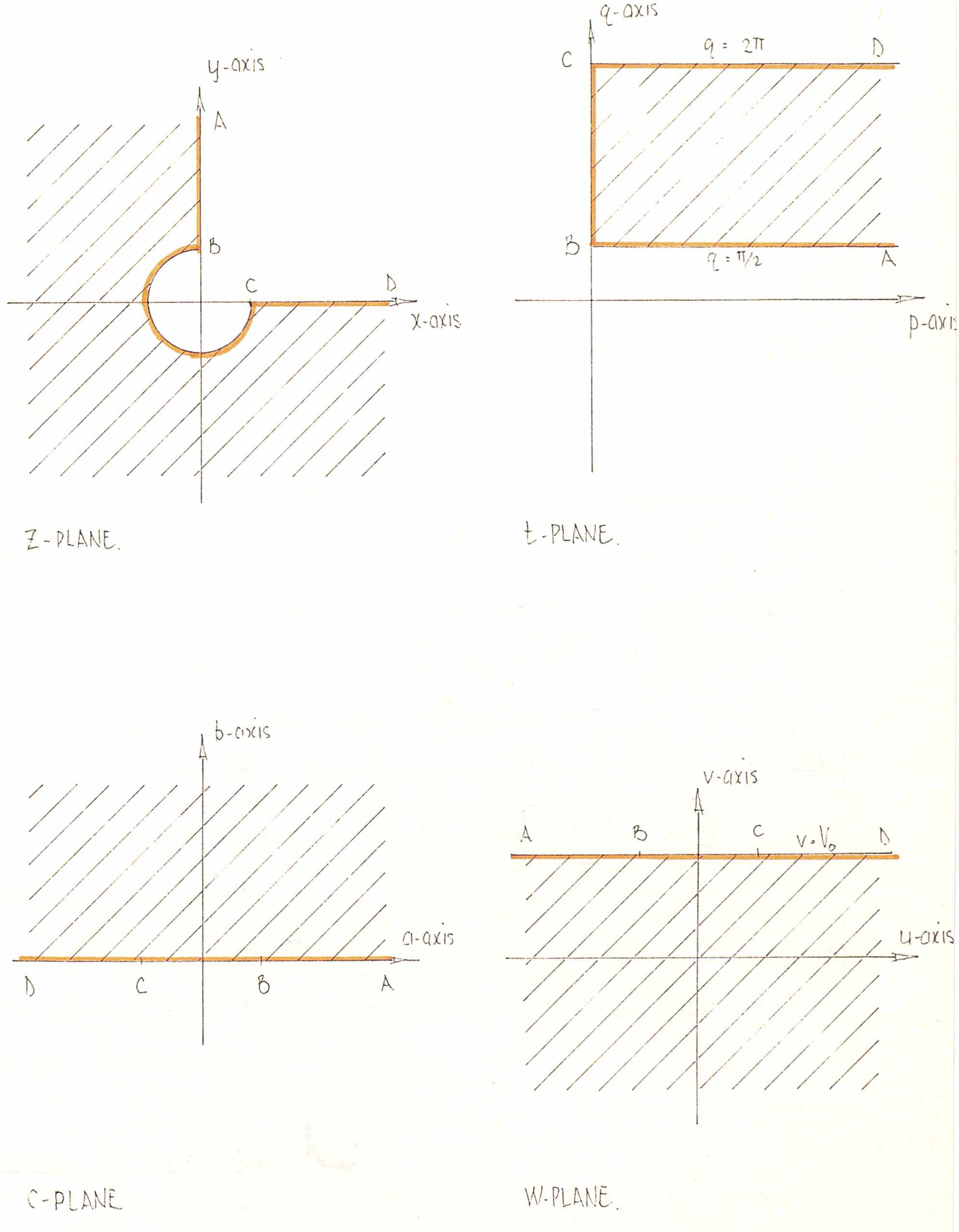
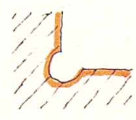


Figure 5.4 Field of 

Therefore $F = \frac{i\pi}{2}$

when $c = -1, \quad t = 2i\pi$

$$2i\pi = B \ln(-1) + \frac{i\pi}{2}$$

Therefore $B = \frac{3}{2}$

Substituting into equation (5.17) gives

$$t = \frac{3}{2} \ln \left[c + \sqrt{c^2 - 1} \right] + \frac{i\pi}{2} \qquad \text{-------------- (5.18)}$$

Electrical Transformation

From equation (5.3) we get

$$c = i\,V_o - w \qquad \text{-------------- (5.19)}$$

Final Transformation

Combining equations (5.19) and (5.18) gives for t;

$$t = \frac{3}{2} \ln \left[i\,V_o - w + \sqrt{(iV_o - w)^2 - 1} \right] + \frac{i\pi}{2}$$

Substituting into equation (5.10) we get

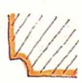
$$z = \exp \left(\frac{3}{2} \ln \left[i\,V_o - w + \sqrt{(iV_o - w)^2 - 1} \right] + \frac{i\pi}{2} \right)$$

$$= \left[i\,V_o - w + \sqrt{(i\,V_o - w)^2 - 1} \right]^{\frac{3}{2}} \exp \frac{i\pi}{2}$$

Therefore $z = i \left[i\,V_o - w + \sqrt{(iV_o - w)^2 - 1} \right]^{\frac{3}{2}}$

$$\text{-------------- (5.20)}$$

Graph 5.4 shows the external field plot obtained from this equation.

Field Strength for 

$$R = \left| \frac{dw}{dz} \right|$$

$$= \left| \frac{dw}{dc} \cdot \frac{dc}{dt} \cdot \frac{dt}{dz} \right|$$

Therefore $R = \left| -1 \,.\; 2\sqrt{c^2 - 1} \,.\; \frac{1}{z} \right|$

To find c in terms of z we refer to equation (5.13)

$$t = \frac{1}{2} \ln \left[c + \sqrt{c^2 - 1} \right]$$

$$= \frac{1}{2} \cosh^{-1} c$$

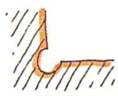
$$= \frac{1}{2} \cosh^{-1} c$$

therefore $c = \cosh(2t)$

and since $t = \ln z$

$$c = \cosh (2 \ln z)$$

Therefore $R = \left| \dfrac{2 \sinh (2 \ln z)}{z} \right|$ ------------ (5.21)

Field Strength for 

$$R = \left| \frac{dw}{dz} \right|$$

$$= \left| \frac{dw}{dc} \cdot \frac{dc}{dt} \cdot \frac{dt}{dz} \right|$$

$$= \left| -1 \cdot \frac{2}{3} \sqrt{c^2 - 1} \cdot \frac{1}{z} \right| \quad \text{------------ (5.22)}$$

To find c in terms of z we use equation (5.18)

$$t = \frac{3}{2} \ln \left[c + \sqrt{c^2 - 1} \right] + \frac{i\pi}{2}$$

$$\frac{2}{3} \left(t - \frac{i\pi}{2}\right) = \cosh^{-1} c$$

Therefore $c = \cosh\left[\frac{2}{3} \left(t - \frac{i\pi}{2}\right)\right]$

But $t = \ln z$

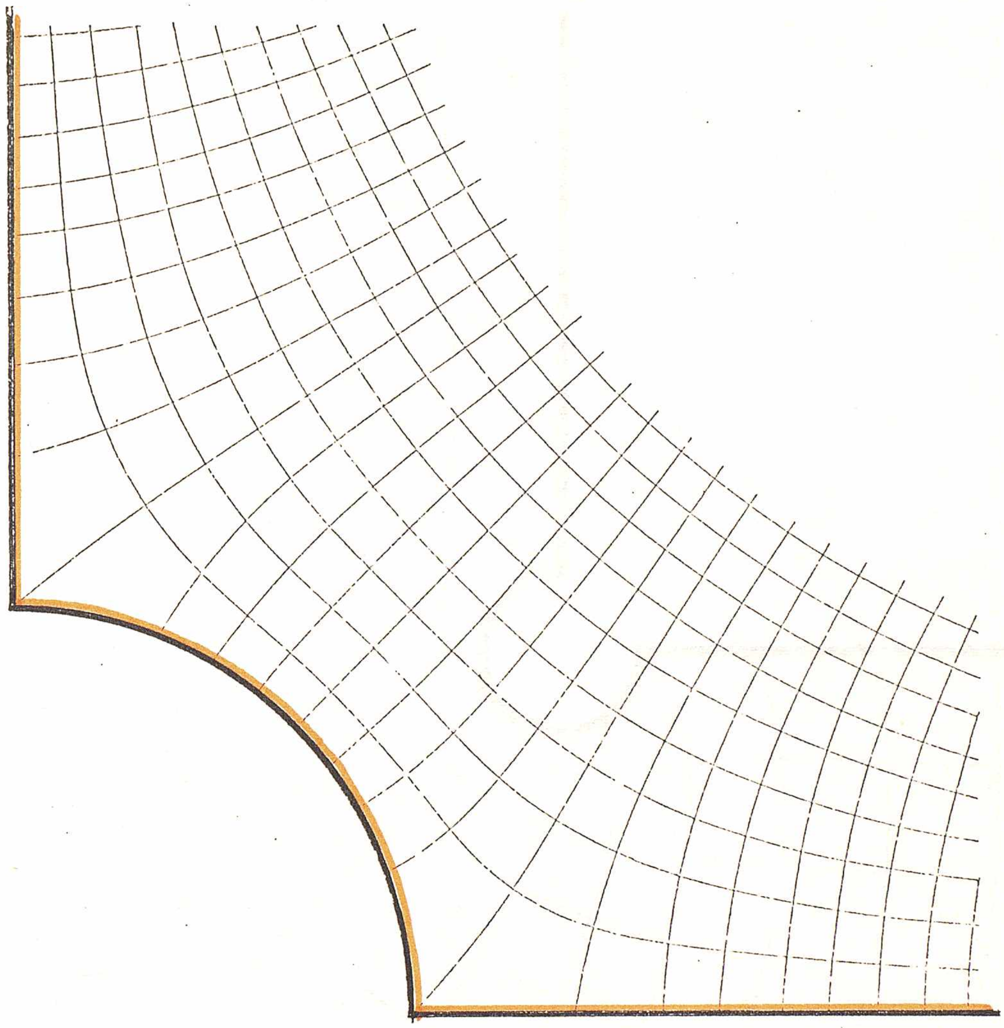
Therefore $c = \cosh\left[\frac{2}{3} \left(\ln z - \frac{i\pi}{2}\right)\right]$

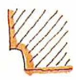
Substituting c into equation (5.22) gives

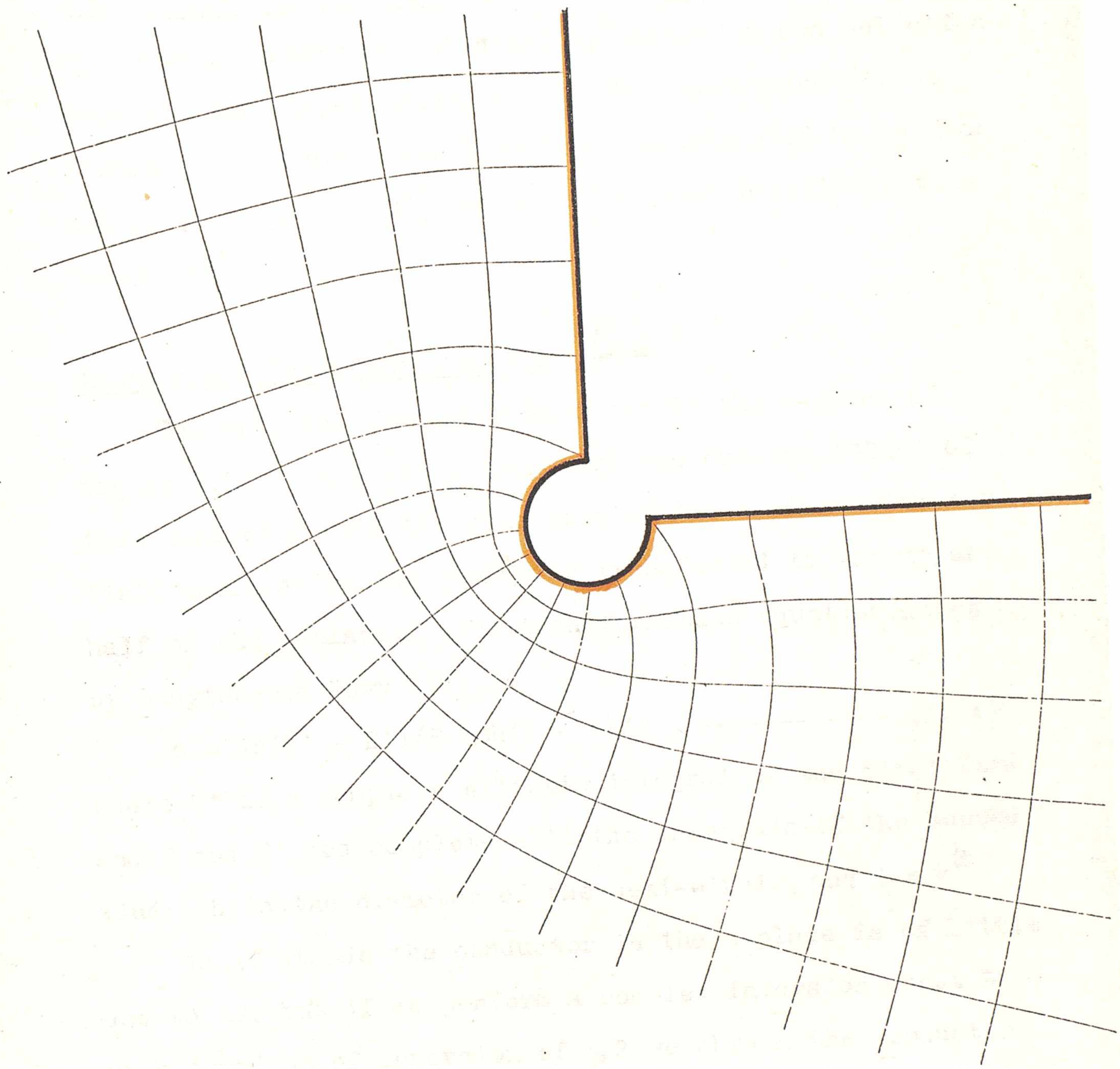
$$R = \frac{2}{3} \left| \frac{\sqrt{\cosh^2\left[\frac{2}{3} \left(\ln z - \frac{i\pi}{2}\right)\right] - 1}}{z} \right|$$

$$\text{Therefore} \quad R = \frac{2}{3}\left|\frac{\sinh\left[\frac{2}{3}\left(\ln z - \frac{j\pi}{2}\right)\right]}{z}\right| \quad \text{---------- (5.23)}$$

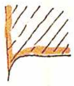
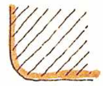
Equations (5.21) and (5.23) give us the required field strength equations for the two corners.



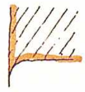
Graph 5.3 Field Plot for 



Graph 5.4 Field Plot for 

5.3 Analysis of Corners  and 

These corners are obtained by inversion and selection from similar shaped conductors. The geometrical transformations in both cases have been investigated by Langton and Davy [15], [16], [17] and will be given briefly in this analysis.

Geometrical Transformation for 

The original conductor is shown in the z-plane of figure 5.5. We have to transform the outline ABDEA' of this conductor onto the real axis of the c-plane with the field enclosed by the conductor transformed to the upper half of the c-plane. The transformation equation given by Langton and Davy [15] is

$$z = ih(K' - E')/E + h(i - 1)/2 \qquad \text{------------} \qquad (5.24)$$

where K' is a complete elliptic integral of the first kind and E and E' are complete elliptic integrals of the second kind; h is the diameter of the semi-circle, and $c = k^2$.

As it stands the conductor in the z-plane is of little use to us, but if we perform a complex inversion about D with a radius of inversion of h/2 we obtain the conductor DBAED' in the z_1-plane. The internal field in the z-plane is transformed to the area below the conductor in the z_1-plane. The inversion equation is

$$z \cdot z_1 = h^2/4 \qquad \text{------------} \qquad (5.25)$$

From the z_1-plane we see that the desired corner shape is given by the figure DBAPG where APG is the negative imaginary axis of the z_1-plane. By substituting equation (5.24) into equation (5.25) we get

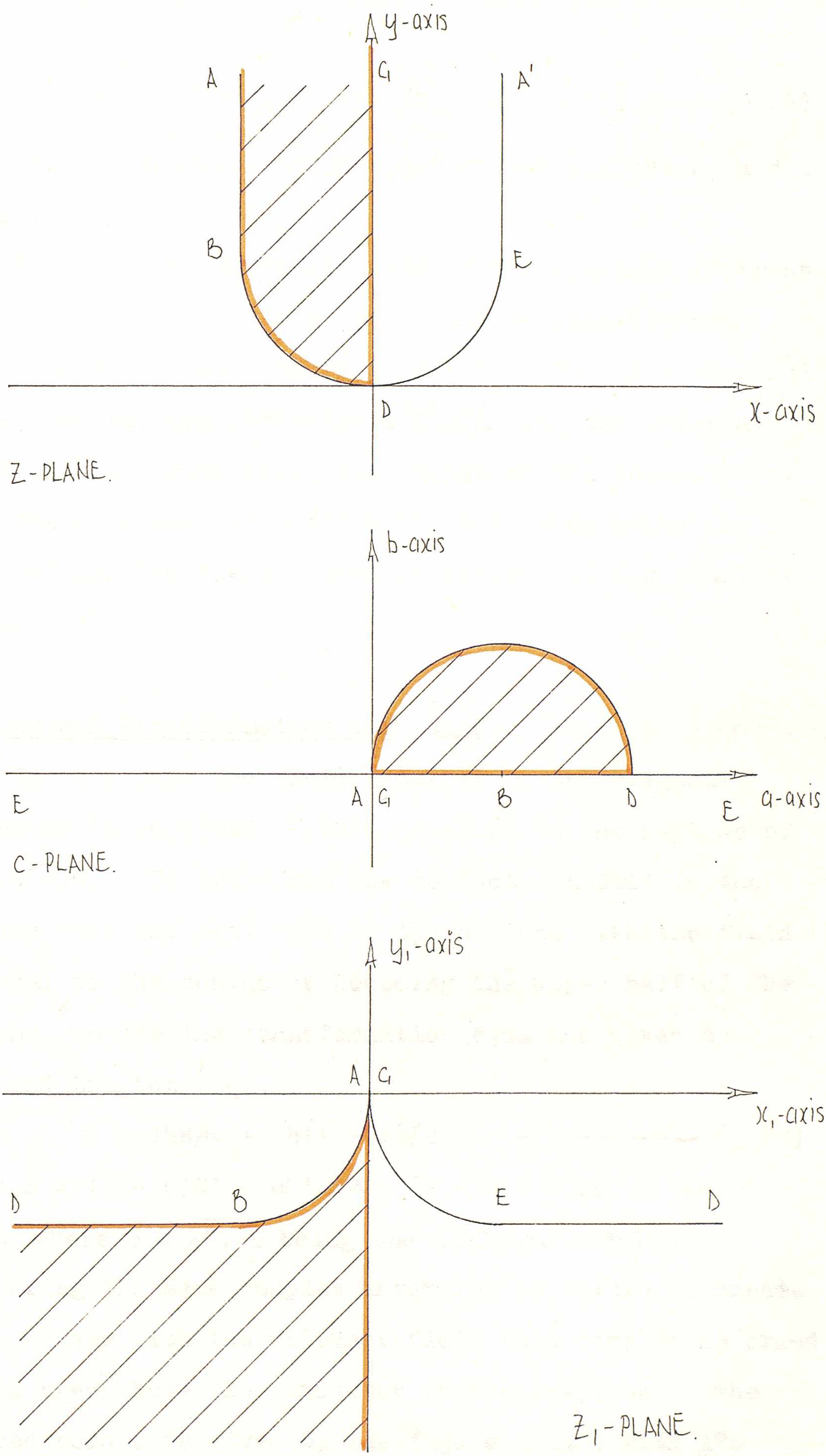
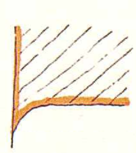


Figure 5.5 Geometrical Transformation for 

$$z_1 = h/\left[4i\left[(K' - E')/E\right] + 2(i - 1)\right] \quad \text{----} \quad (5.26)$$

which is the transformation equation between the z_1 and c planes.

The negative imaginary axis of the z_1-plane is transformed to the semi-circle APG in the c-plane between c = 0 and c = 2 with centre at c = 1. Thus the required corner becomes the semi-circle ABDPA with the internal field of the corner being the inside of the semi-circle.

The relevant conductor outline in each plane is coloured and the field shaded in figure 5.5 for clarification.

Geometrical Transformation for 

In this case the field external to the original conductor is required. This is shown in the z-plane of figure 5.6. To transform the conductor ABDEA' in the z-plane onto the real axis of the c-plane with the field external to the conductor becoming the upper half of the c-plane, we use the transformation equation given by Davy and Langton [16].

$$z = ih\beta/\alpha + h(1 + i)/2 \quad \text{-------------} \quad (5.27)$$

where $\beta = (2 - c)E' - cK'$, $\alpha = (2 - c)E - 2c'K$ and $c' = 1 - c$, where $c = k^2$, k being the elliptic modulus.

Using the same complex inversion as before we create the z_1-plane with the relevant field this time transformed to the area above the conductor in the z_1-plane. The desired corner is given by the figure GPAED, with APG being the positive imaginary axis of the z_1-plane. Combining equations (5.25) and (5.27) we get

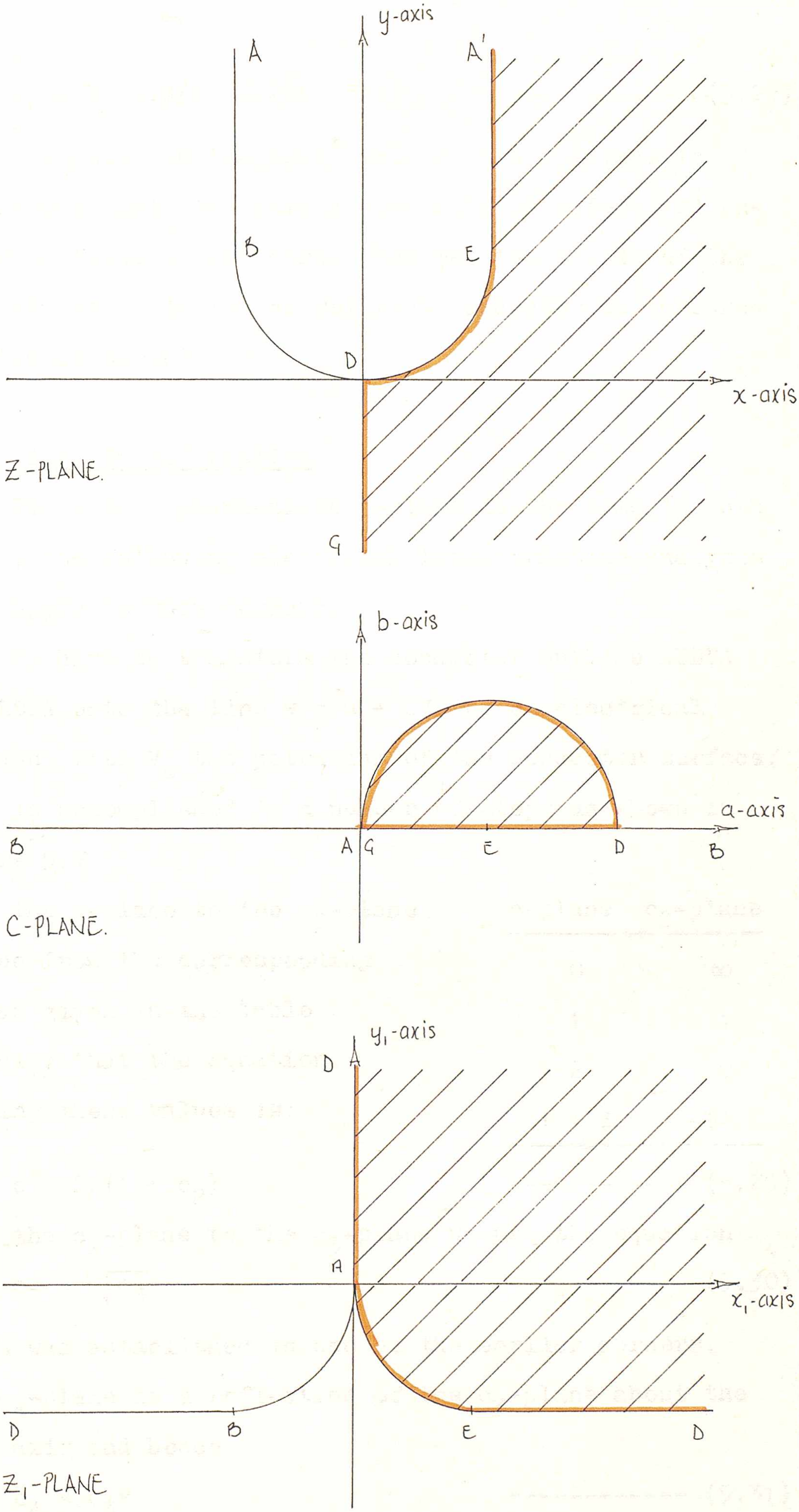
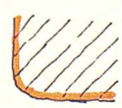


Figure 5.6 Geometrical Transformation for 

$$z_1 = h/\left[4i\beta/\alpha + 2(1 + i)\right] \qquad \text{------------} \quad (5.28)$$

The positive imaginary axis of the z_1-plane is transformed into the same semi-circle as before and the internal field of the corner becomes the inside of the semi-circle. Again the relevant conductor is coloured and fields shaded.

Electrical Transformation

Since the intermediate c-plane is the same in both cases, the following electrical transformation analysis will apply to both corners.

We have to transform the conductor outline ABDPA or AEDPA onto the line $w = u + iV_o$ on an electrical w-plane, with V_o the potential of the conductor surface. This is accomplished in a number of steps as shown in figure 5.7

From the c-plane to the c_2-plane we see from the corresponding values given in the table opposite that the equation fitting these values is:

c-plane	c_2-plane
0	∞
1	1
2	0
1 + i	-1

$$c = 2/(1 + c_2) \qquad \text{------------} \quad (5.29)$$

From the c_2-plane to the c_3-plane we use the equation

$$c_2 = \sqrt{c_3} \qquad \text{------------} \quad (5.30)$$

which was established in one of the earlier corners.

The c_4-plane is a reflection of the c_3-plane about the real axis and hence

$$c_4 = c_3^* \qquad \text{------------} \quad (5.31)$$

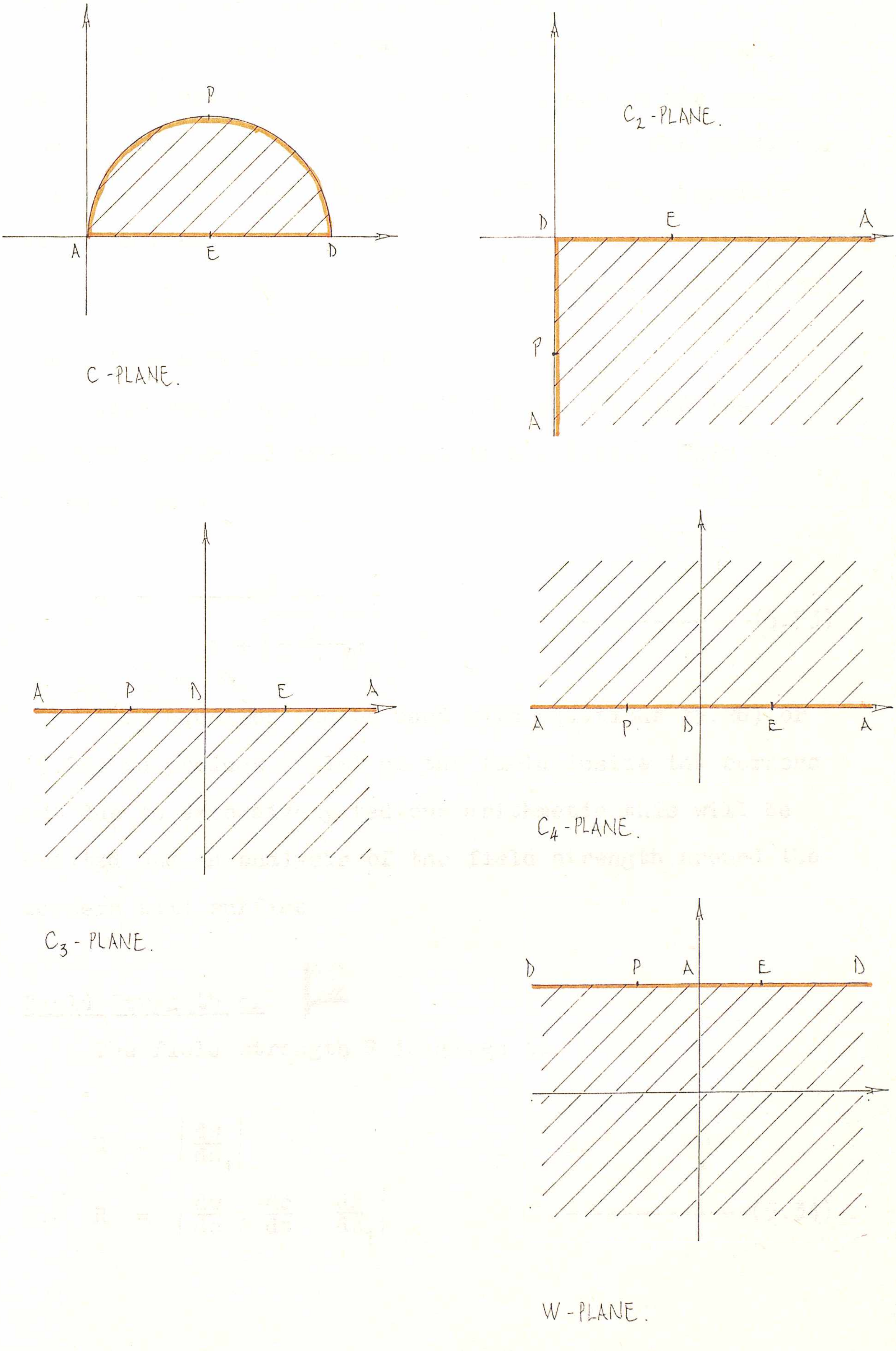


Figure 5.7 Electrical Transformation

where c_3^* is the complex conjugate of c_3.

To obtain the final transformation $c_4 \rightarrow$ w-plane, we must invert c_4 about the point D since in the conductor plane z_1 the point D appears at ∞. The conductor potential has also to be raised to V_o. The transformation becomes

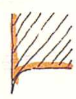
$$w = \frac{A}{c_4} + iV_o \qquad \text{------------ (5.32)}$$

where A is a real constant.

With equations (5.29) $\rightarrow$ (5.32) we can find the overall electrical transformation $c = f(w)$. This is found to be

$$c = \frac{2}{1 + \sqrt{\dfrac{A}{w - iV_o}}} \qquad \text{------------- (5.33)}$$

This equation can be used with equations (5.26) or (5.28) to produce a plot of the field inside the corners but due to excessively tedious arithmetic this will be omitted and an analysis of the field strength around the corners will suffice.

<u>Field Strength of</u> 

The field strength R is given as

$$R = \left| \frac{dw}{dz_1} \right|$$

$$\text{i.e. } R = \left| \frac{dw}{dc} \cdot \frac{dc}{dz} \cdot \frac{dz}{dz_1} \right| \qquad \text{------------- (5.34)}$$

From equation (5.33) we get

$$w = \frac{Ac^2}{(2 - c)^2} + iV$$

$$\frac{dw}{dc} = A\left[(2 - c)^2 \cdot 2c + c^2 \cdot 2(2 - c)\right] / (2 - c)^4$$

Therefore $$\frac{dw}{dc} = \frac{4Ac}{(2 - c)^3} \qquad (5.35)$$

From equation (5.24)

$$z = ih(K' - E')/E + h(i - 1)/2$$

$$\frac{dz}{dc} = ih \frac{d}{dc}\left[\frac{K' - E'}{E}\right]$$

this is given by Cayley [18] as

$$\frac{dz}{dc} = -\frac{i\pi h}{4cE^2} \qquad (5.36)$$

The inversion equation (5.25) gives

$$z = \frac{h^2}{4z_1}$$

$$\frac{dz}{dz_1} = \frac{-h^2}{4z_1^2} \qquad (5.37)$$

Substituting equations (5.35), (5.36) and (5.37) into equation (5.34) gives

$$R = \left|\frac{4Ac}{(2 - c)^3} \cdot \frac{-4cE^2}{i\pi h} \cdot \frac{-h^2}{4z_1^2}\right|$$

$$= \frac{4Ah}{\pi}\left|\frac{c^2 E^2}{(2 - c)^3 z_1^2}\right|$$

In terms of z_1/h we get that

$$R = \frac{4A}{\pi h} \left| \frac{c^2 \; E^2}{(2 - c)^3 \; (z_1/h)^2} \right| \qquad \text{------------} \quad (5.38)$$

With this equation we can find the field strength along the surface of the conductor or at any point in the field. We will confine our analysis to the conductor surface DAPG in the z_1-plane of figure 5.5. Table 5.1 shows the field strength values for the corresponding c and z_1/h values.

Point on Conductor	c	z_1/h	$R \frac{\pi h}{4A}$
A	0	0	0
	0.25	- 0.0802 - 0.1823i	0.628
	0.50	- 0.129 - 0.219i	2.092
	0.75	- 0.1765 - 0.2588i	4.788
	0.9046	- 0.2172 - 0.248i	6.95
	0.97	- 0.239 - 0.2498i	7.795
B	1.00	- 0.25 - 0.25i	8.0
	1.03	- 0.264 - 0.25i	8.388
	1.137	- 0.314 - 0.25i	9.756
	1.334	- 0.47 - 0.25i	13.34
	1.703	- 1.331 - 0.25i	38.28
D, G	2.00	∞	∞
	0.11 + 0.458i	- 0.245i	0.153
	0.293+ 0.707i	- 0.305i	1.254
P	1 + i	- 0.441i	6.962
	1.588+ 0.81i	- 0.092i	17.37
	1.82 + 0.458i	- 1.7i	27.06

Table 5.1 Field Strength values for 

A graph showing the effect of field strength variations around the corner was made and this is reproduced in Graph 5.5.

Field Strength of 

The only difference in this case lies in the value of dz/dc

$$z = ih\beta/\alpha + h(1 + i)/2$$

$$\frac{dz}{dc} = ih\frac{d}{dc}\,\beta/\alpha$$

This is given by Cayley [18] as

$$\frac{dz}{dc} = -\frac{3c\pi ih}{4\alpha^2} \qquad \text{------------- (5.39)}$$

Using equation (5.39) with equations (5.35) and (5.37) gives

$$R = \left| \frac{4Ac}{(2 - c)^3} \cdot -\frac{4\alpha^2}{3c\pi ih} \cdot -\frac{h^2}{4z_1{}^2} \right|$$

$$= \frac{4Ah}{3\pi} \left| \frac{\alpha^2}{(2 - c)^3\, z_1{}^2} \right|$$

in terms of z_1/h we get

$$R = \frac{4A}{3\pi h} \left| \frac{\alpha^2}{(2 - c)^3\, (z_1/h)^2} \right|$$

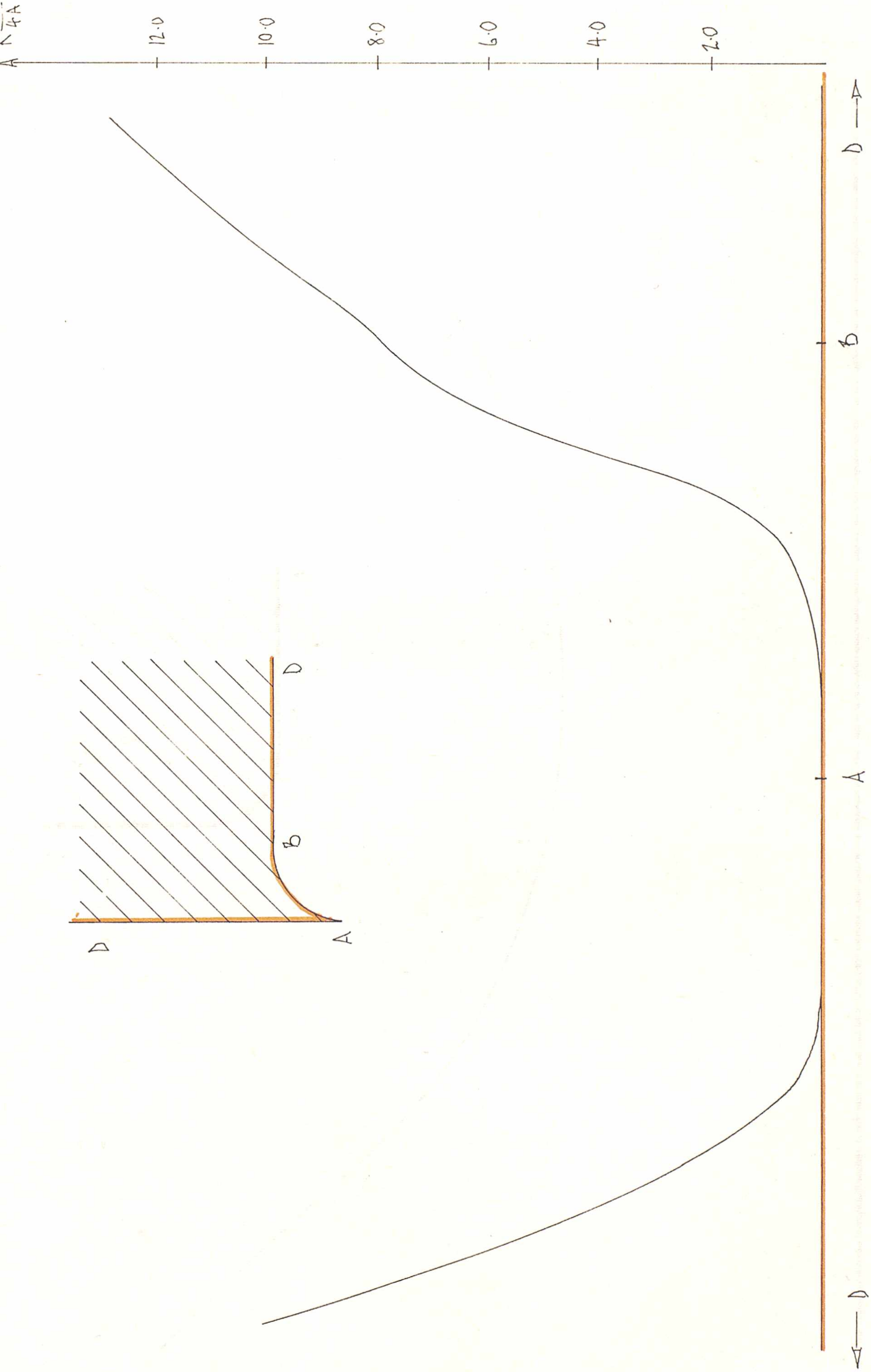
------------- (5.40)

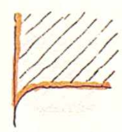
Again we find the field strength along the surface of the conductor GPAED in the z_1-plane of figure 5.6. Table 5.2 shows the field strength values. Since the figure is symmetrical the values along GA will be similar to those along ED and hence are omitted.

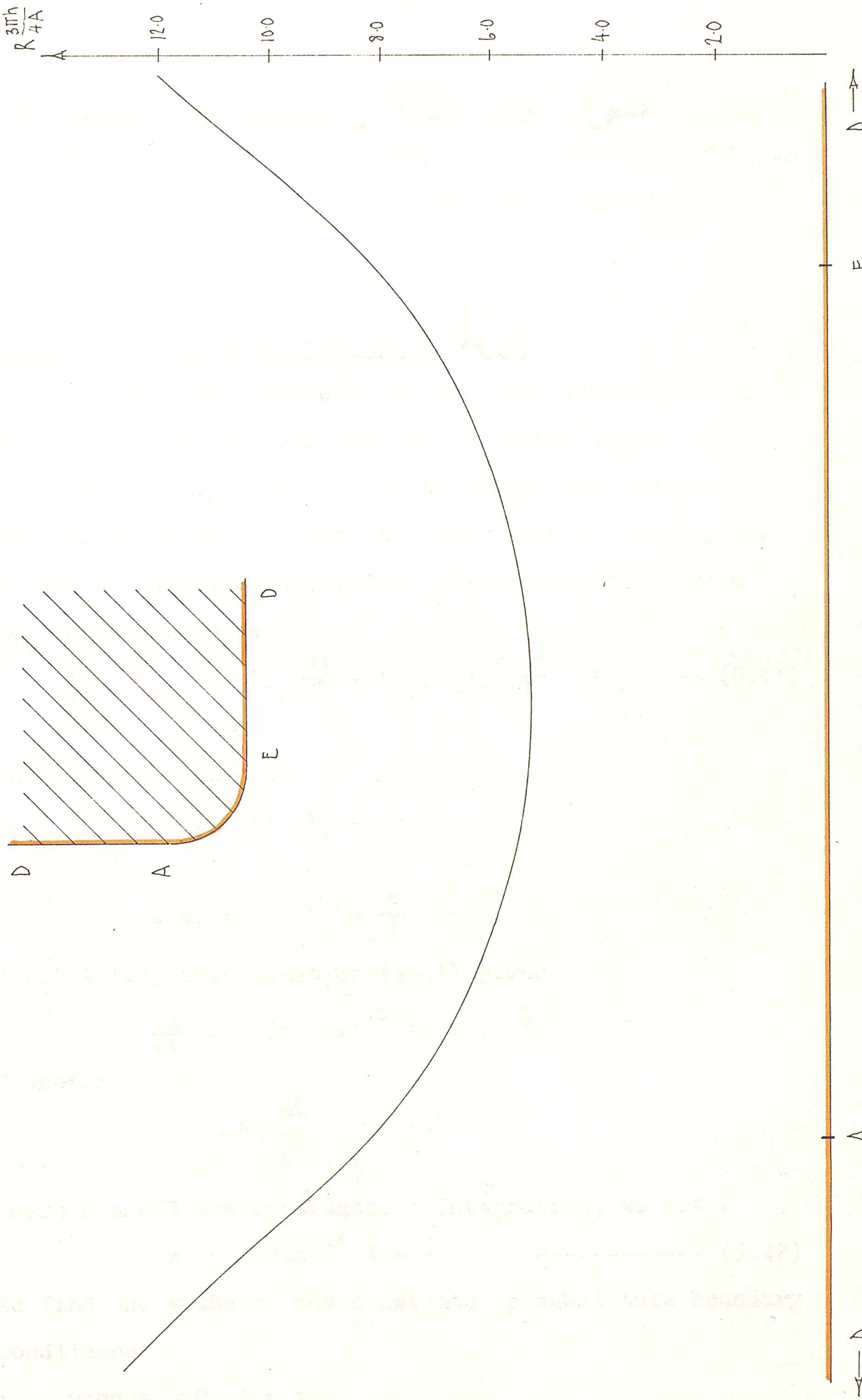
Graph 5.6 shows the effect of the field strength variation around the corner.

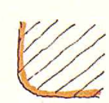
Point on Conductor	c	z_1/h	R $\frac{3\pi h}{4A}$
A	0	0	8.0
	0.50	0.0026 - 0.036i	6.705
	0.667	0.0156 - 0.087i	5.85
	0.75	0.0353 - 0.128i	5.494
	0.80	0.0568 - 0.1586i	5.333
	0.8284	0.0738 - 0.1773i	5.2
	0.85	0.0881 - 0.19i	5.35
	0.90	0.1315 - 0.22i	5.604
	0.9532	0.1923 - 0.243i	6.166
E	1.0	0.25 - 0.25i	8.0
	1.03	0.285 - 0.25i	9.58
	1.137	0.405 - 0.25i	14.19
	1.703	2.1 - 0.25i	72.17
D, G	2.0	∞	∞

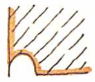
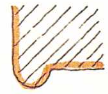
Table 5.2 Field Strength values for 



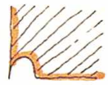
Graph 5.5 Field Strength Variation around 



<u>Graph 5.6</u> Field Strength Variation around 

5.4 Analysis of Corners  and 

Those corners can be obtained by inversions of known conductor shapes, one of which has been studied in Chapter 4.

Geometrical Transformation for 

The original conductor is shown in the z-plane of figure 5.8 with the relevant field inside the semi-infinite rectangle ABCD. To transform the perimeter ABCD of the conductor onto the real axis of the t-plane we use the Schwarz-Christoffel transformation. With two corners we have

$$\frac{dz}{dt} = A\,(t - t_1)^{\frac{\alpha_1}{\pi} - 1}\,(t - t_2)^{\frac{\alpha_2}{\pi} - 1} \quad \text{------} \quad (5.41)$$

The mapping table is

point B $t_1 = -1$ $\alpha_1 = \frac{\pi}{2}$

C $t_1 = 1$ $\alpha_2 = \frac{\pi}{2}$

Substituting into equation (5.41) gives

$$\frac{dz}{dt} = A\,(t + 1)^{-\frac{1}{2}}\,(t - 1)^{-\frac{1}{2}}$$

Therefore

$$z = A\int \frac{dt}{\sqrt{t^2 - 1}} + B$$

where A and B are constants. Integrating, we get

$$z = A\cosh^{-1} t + B \quad \text{------------} \quad (5.42)$$

To find the value of the constants we substitute boundary conditions.

1. when $z = 0$, $t = 1$

Therefore $B = 0$

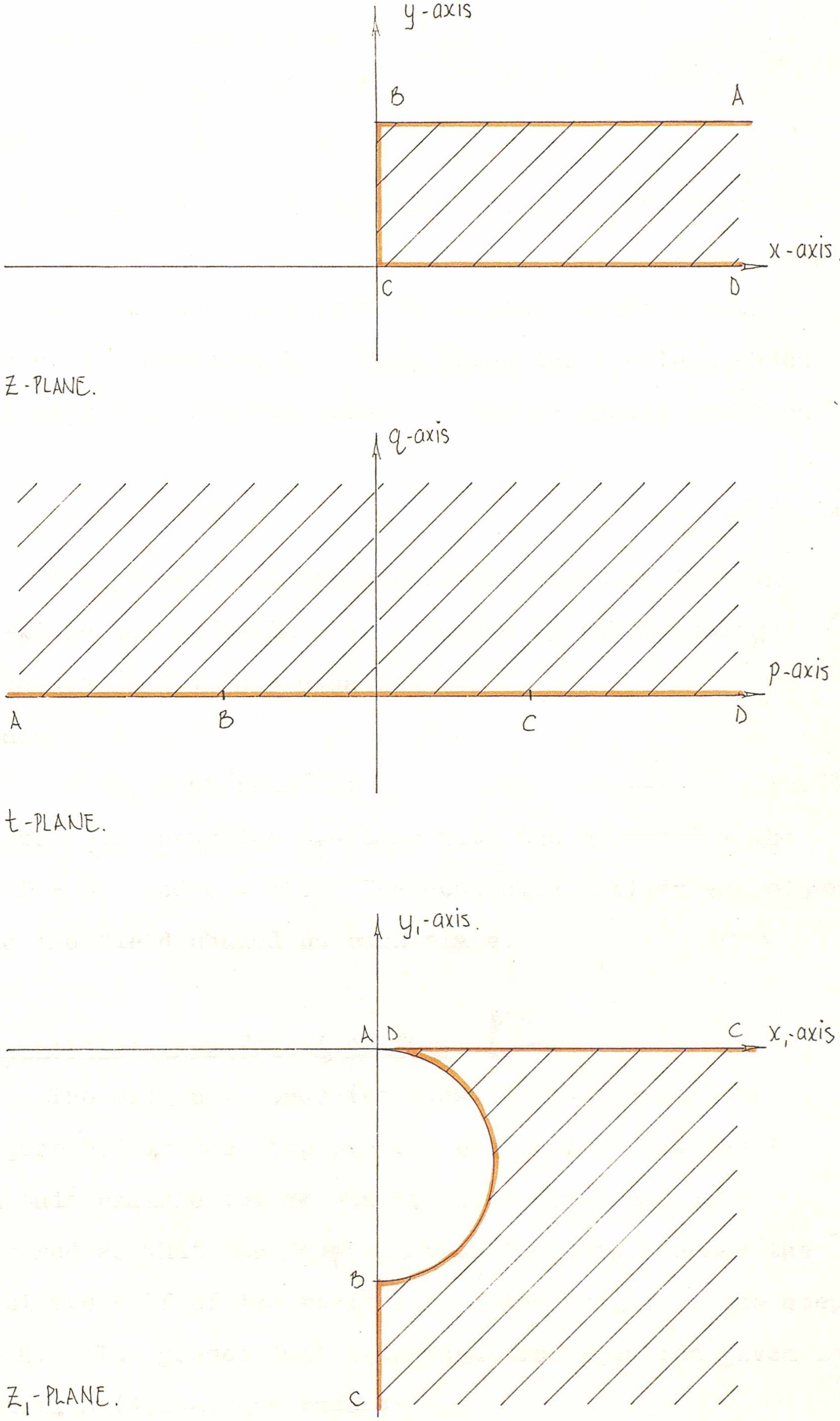
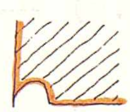


Figure 5.8 Geometrical Transformation for 

2. when $z = ih$, $t = -1$

$$ih = Ai\pi + 0$$

Therefore $A = \frac{h}{\pi}$

Substituting back into equation (5.42) we get that

$$z = \frac{h}{\pi} \cosh^{-1} t \qquad \text{------------- (5.43)}$$

We now have to invert the z-plane about C with radius of inversion h. This gives the z_1-plane which provides the required corner. The inversion equation is

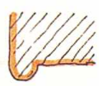
$$z . z_1 = h^2 \qquad \text{-------------- (5.44)}$$

The geometrical transformation equation from the z_1-plane to the t-plane is obtained by substituting equation (5.43) into equation (5.44).

Hence

$$z_1 = h\pi/\cosh^{-1} t \qquad \text{-------------- (5.45)}$$

Figure 5.8 shows the z_1-plane with the points B = ih; A, D = 0; and C = ∞. The conductor outline is coloured and the field shaded on each plane.

Geometrical Transformation for 

The original conductor shown in the z-plane of figure 5.9 is the step problem examined in Chapter 4. In this example the origin of the z-plane has been lowered so that the lower part of the step becomes the positive half of the real axis. The height of the step is h. The geometrical transformation equation given in equation (4.20a) now becomes

$$z = \frac{h}{\pi} \left[\sqrt{t^2 - 1} + \cosh^{-1} t \right] \qquad \text{-------- (5.46)}$$

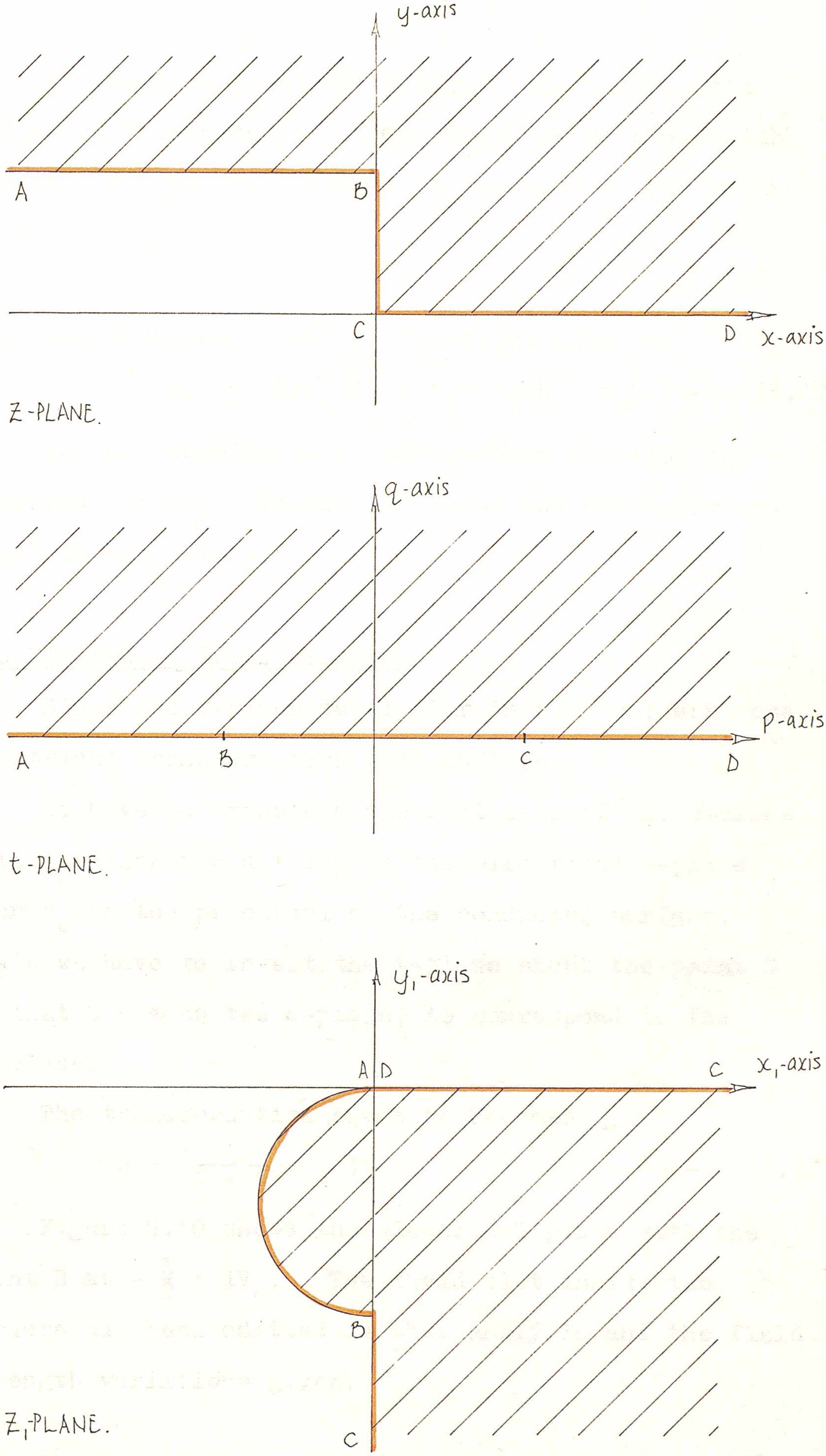
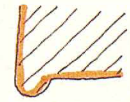


Figure 5.9 Geometrical Transformation for 

The z-plane is inverted about C as before with radius of inversion h. This gives the z_1 plane with the semi-circle now reversed as required. The inversion equation is

$$z \cdot z_1 = h^2$$

and when equation (5.46) is substituted we get

$$z_1 = h\pi / \left[\sqrt{t^2 - 1} + \cosh^{-1} t \right] \quad \text{------} \quad (5.47)$$

This is the geometrical transformation equation for the required corner. Figure 5.9 shows the conductor and field in each plane.

The Electrical Transformation

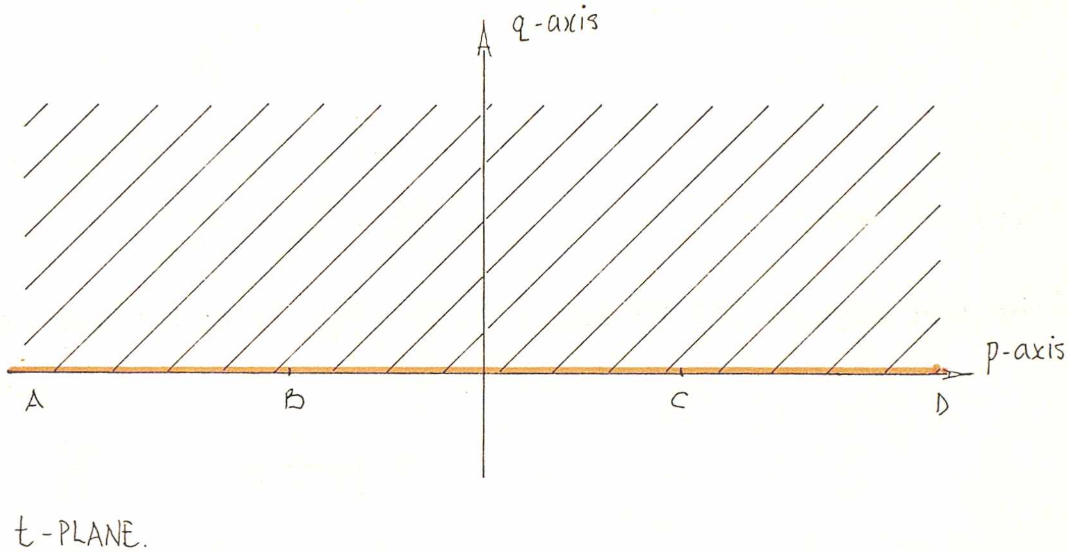
Since the t-plane is similar in both corners, one electrical transformation will suffice.

We have to transform the real axis of the t-plane onto the line $w = u + iV_o$ in the electrical w-plane where V_o is the potential of the conductor surface. Again we have to invert the t-plane about the point C so that $C = \infty$ on the w-plane, to correspond to the z_1-plane.

The transformation equation becomes

$$w = \frac{A}{t - 1} + iV_o \quad \text{-------------} \quad (5.48)$$

Figure 5.10 shows the electrical plane with the point B at $-\frac{A}{2} + iV_o$. The field plot inside the corners has been omitted in this analysis and the field strength variations given.



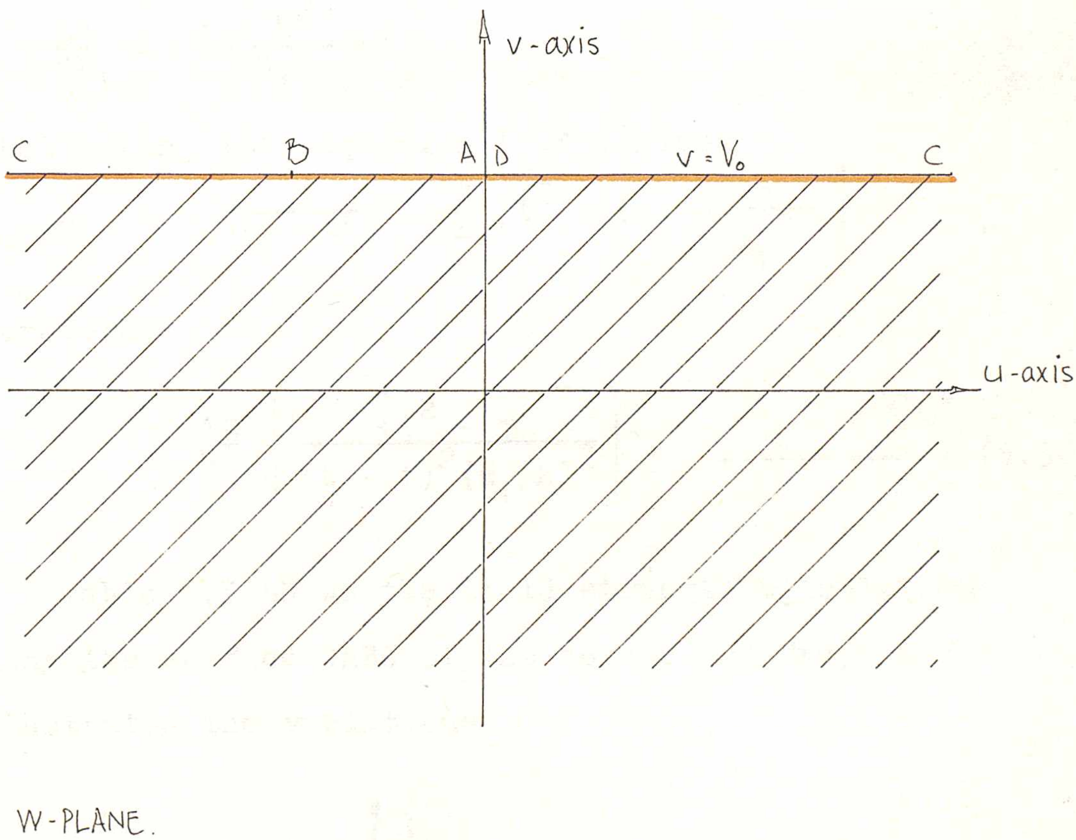
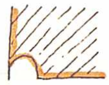


Figure 5.10 Electrical Transformation.

Field Strength of 

The field strength R is given as

$$R = \left| \frac{dw}{dz_1} \right|$$

This can be written as

$$R = \left| \frac{dw}{dt} \cdot \frac{dt}{dz} \cdot \frac{dz}{dz_1} \right| \qquad \text{------------- (5.49)}$$

From equations (5.48), (5.43) and (5.44) we get respectively

$$\frac{dw}{dt} = - \frac{A}{(t - 1)^2}$$

$$\frac{dz}{dt} = \frac{h}{\pi} \cdot \frac{1}{\sqrt{t^2 - 1}}$$

$$\frac{dz}{dz_1} = - \frac{h^2}{{z_1}^2}$$

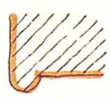
Substituting into equation (5.49) gives

$$R = \left| \frac{A}{(t - 1)^2} \cdot \frac{\pi}{h} \sqrt{t^2 - 1} \cdot - \frac{h^2}{{z_1}^2} \right|$$

Therefore

$$R = \frac{A\pi}{h} \left| \frac{\sqrt{t^2 - 1}}{(t - 1)^2 (z_1/h)^2} \right| \qquad \text{--------------- (5.50)}$$

Table 5.3 shows the field strength calculations along the surface CABC of the corner and Graph 5.7 illustrates the variations.

Field Strength of 

From equations (5.48), (5.46) and (5.44) we get respectively

$$\frac{dw}{dz} = - \frac{A}{(t - 1)^2}$$

$$\frac{dz}{dt} = \frac{h}{\pi}\sqrt{\frac{t+1}{t-1}}$$

$$\frac{dz}{dz_1} = -\frac{h^2}{{z_1}^2}$$

Substituting into equation (5.49) gives

$$R = \left| -\frac{A}{(t-1)^2} . \frac{\pi}{h}\sqrt{\frac{t-1}{t+1}} . -\frac{h^2}{{z_1}^2} \right|$$

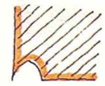
and hence $R = \frac{A\pi}{h} \left| \frac{1}{(t-1)^2 \; (z_1/h)^2} \sqrt{\frac{t-1}{t+1}} \right|$

------------- (5.51)

Table 5.4 gives the field strength calculations along CABC of the z_1-plane and graph 5.8 illustrates the variations.

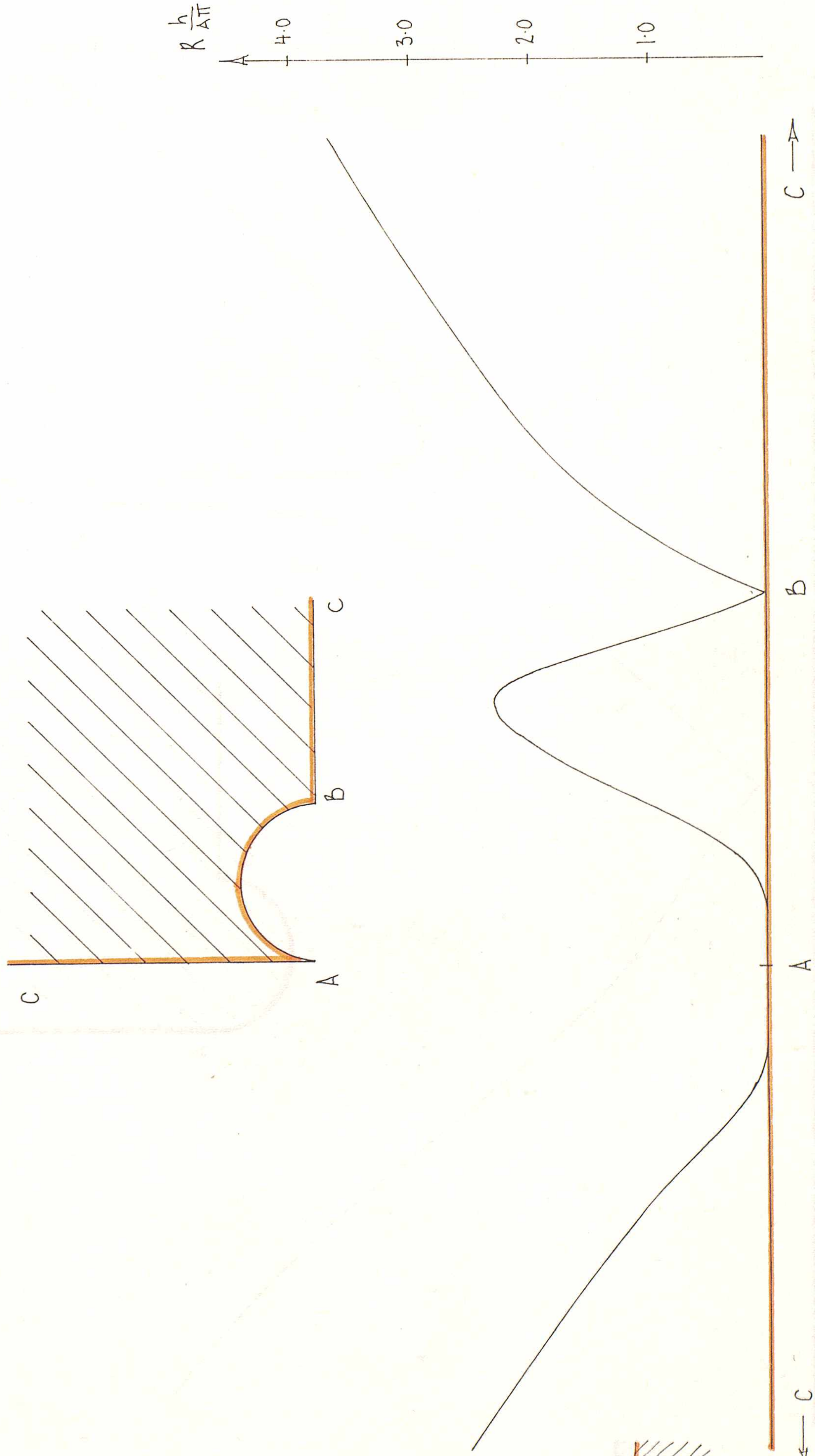
In using equations (5.46) and (5.47) care must be taken in the sign of the real part. The z-plane of figure 5.9 shows that for $-\infty < t < -1$, the real part of z is negative and although equation (5.46) might indicate otherwise, the correct sign must be applied. The reason for this lies in the square root and $\cosh^{-1}$ functions of equation (5.46) which can be positive or negative. This also applies with equation (5.47) where the z_1-plane has a negative real part for $-\infty < t < -1$.

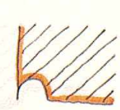
t	z/h	z_1/h	$R \frac{h}{A\pi}$
∞	∞	0	0
100	1.69	0.59	0.287
7.0	0.838	1.19	1.34
5.0	0.73	1.37	1.6
3.0	0.56	1.78	2.2
1.5	0.306	3.27	4.14
1.25	0.22	4.55	5.76
1.1	0.14	7.14	9.0
1.0	0	∞	∞
0.9	0.144i	- 6.94 i	8.8
0.7	0.25i	- 4.0i	5.0
0.5	0.33i	- 3.0i	3.8
0.3	0.4i	- 2.5i	3.12
0.1	0.47i	- 2.13i	2.65
0.0	0.5i	- 2.0i	2.47
- 0.1	0.53i	- 1.89i	2.3
- 0.3	0.6i	- 1.67i	1.98
- 0.5	0.67i	- 1.49i	1.69
- 0.7	0.747i	- 1.34i	1.36
- 0.9	0.85i	- 1.18i	0.87
- 1.0	1.0i	- 1.0i	0
- 1.1	0.14 + i	0.137 - 0.98i	1.05
- 1.25	0.22 + i	0.21 - 0.95i	1.53
- 1.5	0.306 + i	0.28 - 0.91i	1.93
- 3.0	0.56 + i	0.43 - 0.76i	2.3
- 5.0	0.73 + i	0.48 - 0.65i	2.06
- 7.0	0.838 + i	0.49 - 0.59i	1.82
- 100	1.69 + i	0.44 - 0.26i	0.37
- ∞	∞ + i	0	0

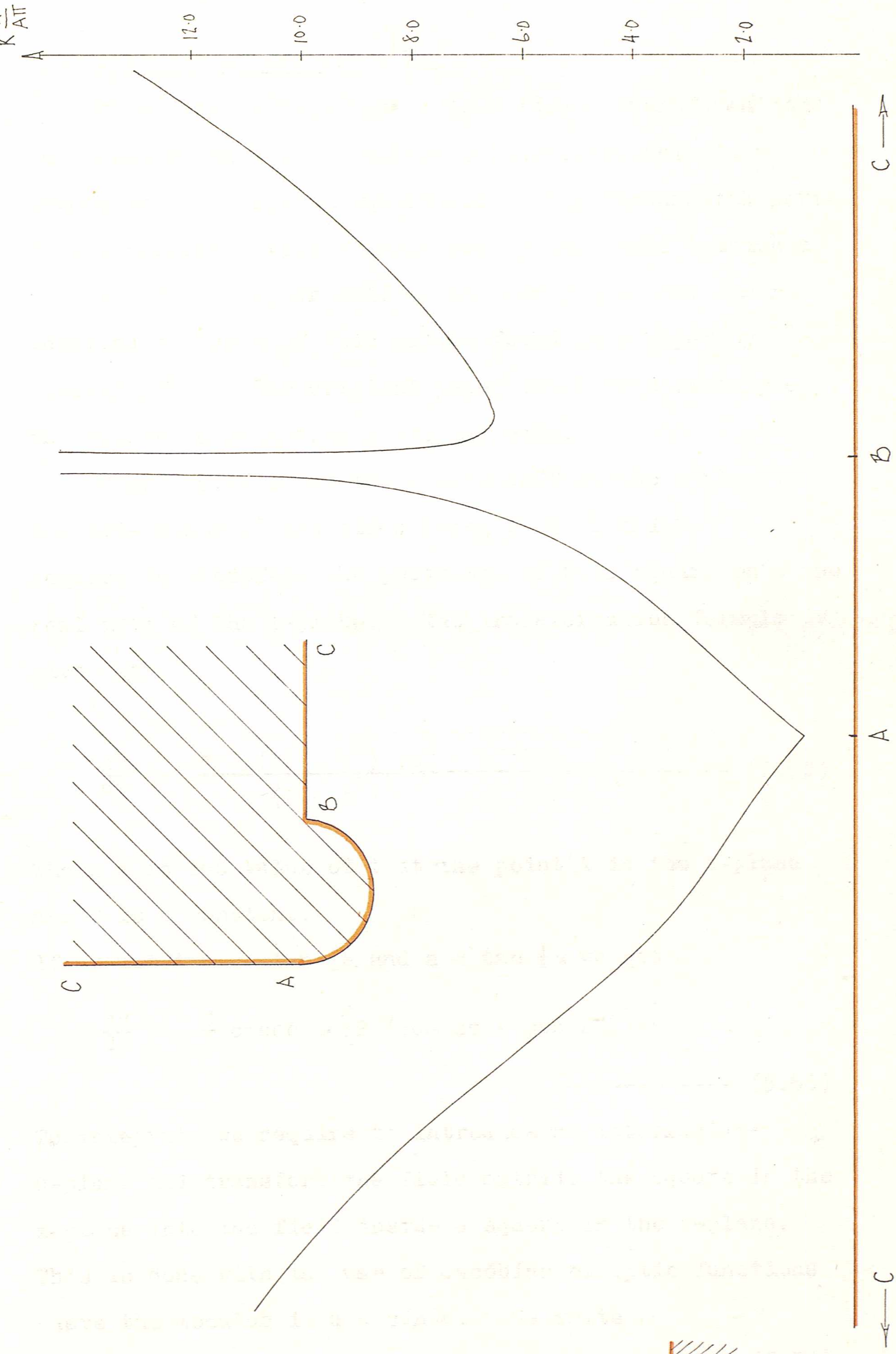
Table 5.3 Field Strength Calculations for 

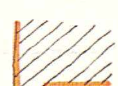
t	z/h	z_1/h	$R \frac{h}{A\pi}$
∞	∞	0	1.0
100	33.52	0.03	1.12
7.0	3.04	0.33	2.21
5.0	2.29	0.437	2.65
3.0	1.47	0.685	3.75
1.5	0.66	1.51	7.8
1.25	0.46	2.17	10.98
1.1	0.29	3.45	18.0
1.0	0	∞	∞
0.9	0.28i	- 3.57i	18.1
0.7	0.48i	- 2.08i	10.64
0.5	0.61i	- 1.64i	8.48
0.3	0.7i	- 1.43i	7.41
0.1	0.78i	- 1.78i	6.79
0.0	0.82i	- 1.22i	6.61
- 0.1	0.85i	- 1.18i	6.46
- 0.3	0.9i	- 1.11i	6.44
- 0.5	0.94i	- 1.06i	6.73
- 0.7	0.97i	- 1.03i	8.08
- 0.9	0.995i	- 1.005i	11.8
- 1.0	1.0i	- 1.0i	∞
- 1.1	- 0.29 + i	- 0.27 - 0.92i	11.1
- 1.25	- 0.46 + i	- 0.38 - 0.825i	7.09
- 1.5	- 0.66 + i	- 0.46 - 0.7i	5.09
- 3.0	- 1.46 + i	- 0.47 - 0.32i	2.73
- 5.0	- 2.29 + i	- 0.37 - 0.16i	2.09
- 7.0	- 3.04 + i	- 0.3 - 0.1i	1.82
-100	- 33.52 + i	- 0.03 - 0.001i	1.1
- ∞	∞ + i	0	1.0

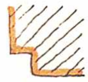
Table 5.4 Field Strength Calculations for 



Graph 5.7 Field Strength Variation for 



Graph 5.8 Field Strength Variation for 

5.5 Analysis of Corner 

This example requires no less than 6 transformations and involves the use of elliptic functions, selection, inversion and the Richmond method. The fundamental part of the problem involves transforming the field outside a square onto the upper half of another plane, and a more detailed analysis of this can be found in a paper by Bickley [19]. The original paper involved a rectangle, the square constituting a special case.

Figure 5.11 shows the square ABCD in the z-plane, the mid-points of the sides being P, Q, L and S. We require to transform the perimeter of this square onto the real axis of the t-plane. The transformation formula is given as

$$\frac{dz}{dt} = \frac{C\sqrt{(t^2 - a^2)(t^2 - a^{-2})}}{(1 + t^2)^2} \qquad \text{------------- (5.52)}$$

Where a is the value of t at the point A in the t-plane and C is a constant.

If we let $t = -\tan\frac{1}{2}s$ and $a = \tan\frac{1}{2}\alpha$ we get

$$\frac{dz}{ds} = \frac{C}{4}\operatorname{cosec}\alpha\left[2(\cos 2s - \cos 2\alpha)\right]^{\frac{1}{2}}$$

------------- (5.53)

To integrate we require to introduce an intermediate u-plane and transform the field outside the square in the z-plane into the field inside a square in the u-plane. This is done with the use of Jacobian elliptic functions where the modulus is $k = \sin\alpha$. We write

$$\sin s = -k\,\text{sn}(u, k) \qquad \text{------------- (5.54)}$$

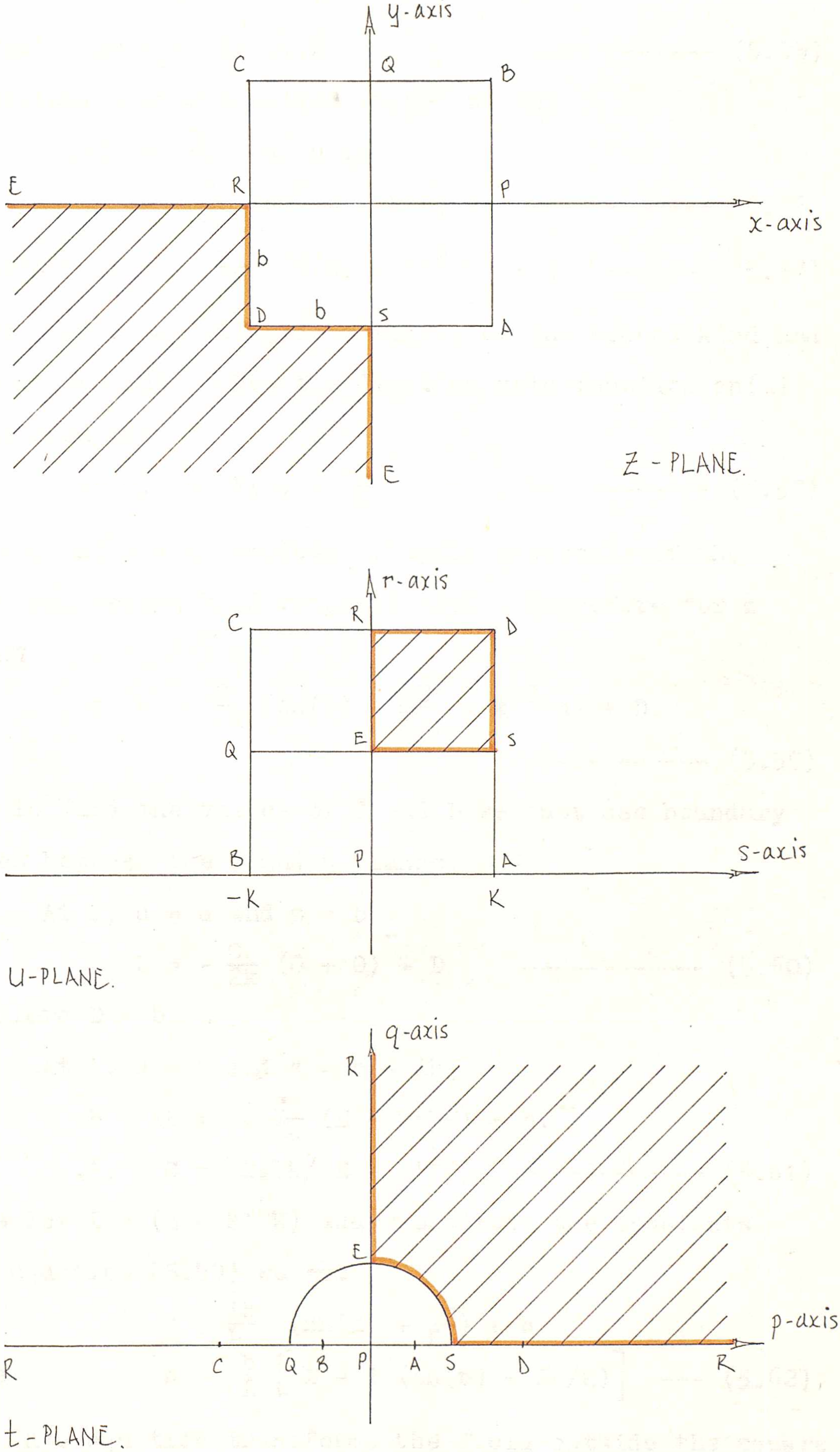


Figure 5.11 Geometrical Transformation

so that $\cos s = \mathrm{dn}\,(u,k)$ ------------ (5.55)

Substituting into equation (5.53) we get

$$dz = -\frac{Ck}{2}\ \mathrm{cn}^2\, u\, du$$

$$= -\frac{C}{2k}\ (\mathrm{dn}^2\, u - k'^2)\, du \qquad \text{------------ (5.56)}$$

Therefore $z = -\frac{C}{2k}\ (E(u) - k'^2\, u) + D$ ------- (5.57)

Where E(u) is an elliptic integral of the second kind and D is a constant. Now the Jacobian zeta function zn(u) is defined as:

$$\mathrm{zn}(u) = E(u) - \frac{uE}{K} \qquad \text{------------ (5.58)}$$

Where K and E are complete elliptic integrals of the first and second kind respectively. Therefore for z we get

$$z = -\frac{C}{2k}\ (\mathrm{zn}(u) + \frac{uE}{K} - k'^2\, u) + D$$

------------ (5.59)

To find the values of C and D we must use boundary values between the z and u planes.

1. At P, u = o and z = b

$$b = -\frac{C}{2k}\ (0 + 0) + D \qquad \text{------------ (5.60)}$$

Therefore D = b.

2. At A, u = K and z = b - ib

$$b - ib = -\frac{C}{2k}\ (E - k'^2 K) + b.$$

$$\therefore \quad C = 2ibk/(E - k'^2 K) \qquad \text{------------ (5.61)}$$

If we let $L = (E - k'^2 K)$ and substitute the constants into equation (5.59) we get

$$z = \frac{ib}{L}\ (\mathrm{zn}(u) + \frac{Lu}{K}) + b$$

hence $z = \frac{b}{L}\left[L - i\ (\mathrm{zn}(u) + Lu/K)\right]$ --- (5.62)

This equation transforms the field outside the square in the z - plane into the field inside the square in the u-plane. The relevant fields are shown in the figure 5.11.

To find the transformation $t = f(u)$ we recall that

$$t = -\tan \tfrac{1}{2}s$$

$$= \frac{\cos s - 1}{\sin s}$$

and since $\cos s = \text{dn}u$ and $\sin s = -k\,\text{sn}u$ we have

$$t = \frac{1 - \text{dn}u}{k\,\text{sn}u} \qquad \text{------------} \quad (5.63)$$

This transforms the perimeter of the square ABDC in the u-plane onto the real axis of the t-plane with the field inside the square transforming to the upper half of the t-plane. Equations (5.62) and (5.63) together provide the geometrical transformation from z to t planes.

From the z-plane in figure 5.11 we can see that the required corner is in fact one quarter of the field and hence we must select this part only for further analysis. The required field is shaded in each of the 3 planes in figure 5.11 with the conductor surface LRDSE coloured for clarification. The interesting part here is the relevant field in the t-plane, since this is one of the corners examined earlier in the thesis (see Section 5.2). We require to transform the outline of the corner RESQR in the t-plane onto the real axis of the c-plane with the field inside the corner becoming the upper half of the c-plane. To do this we refer to Equations (5.10) and (5.13) where, with the relevant notation

$$t = \exp. p$$

$$p = \tfrac{1}{2} \ln \left[c + \sqrt{c^2 - 1} \right]$$

p is an intermediate plane. This gives

$$t = \left[c + \sqrt{c^2 - 1} \right]^{\frac{1}{2}} \qquad \text{------------ (5.64)}$$

Adding equation (5.64) to equations (5.62) and (5.63) completes the geometrical transformation.

Electrical Transformation

We now transform the real axis RESR in the c-plane onto the line $w = u + iV_o$ in the w-plane. We note also that the point E on the original conductor in the z-plane is at infinity and we therefore must invert the c-plane about E to satisfy this requirement. The inversion equation is

$$w = \frac{A}{(c + 1)}$$

Therefore the complete electrical transformation becomes

$$w = \frac{A}{(c + 1)} + iV_o \qquad \text{------------ (5.65)}$$

Figure 5.12 shows the t, c and w planes with the relevant fields shaded.

Field Strength

Field strength R is given as

$$R = \left| \frac{dw}{dz} \right|$$

$$= \left| \frac{dw}{dc} \cdot \frac{dc}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{dz} \right| \qquad \text{------------ (5.66)}$$

From equations (5.62), (5.63), (5.64) and (5.65) we get respectively

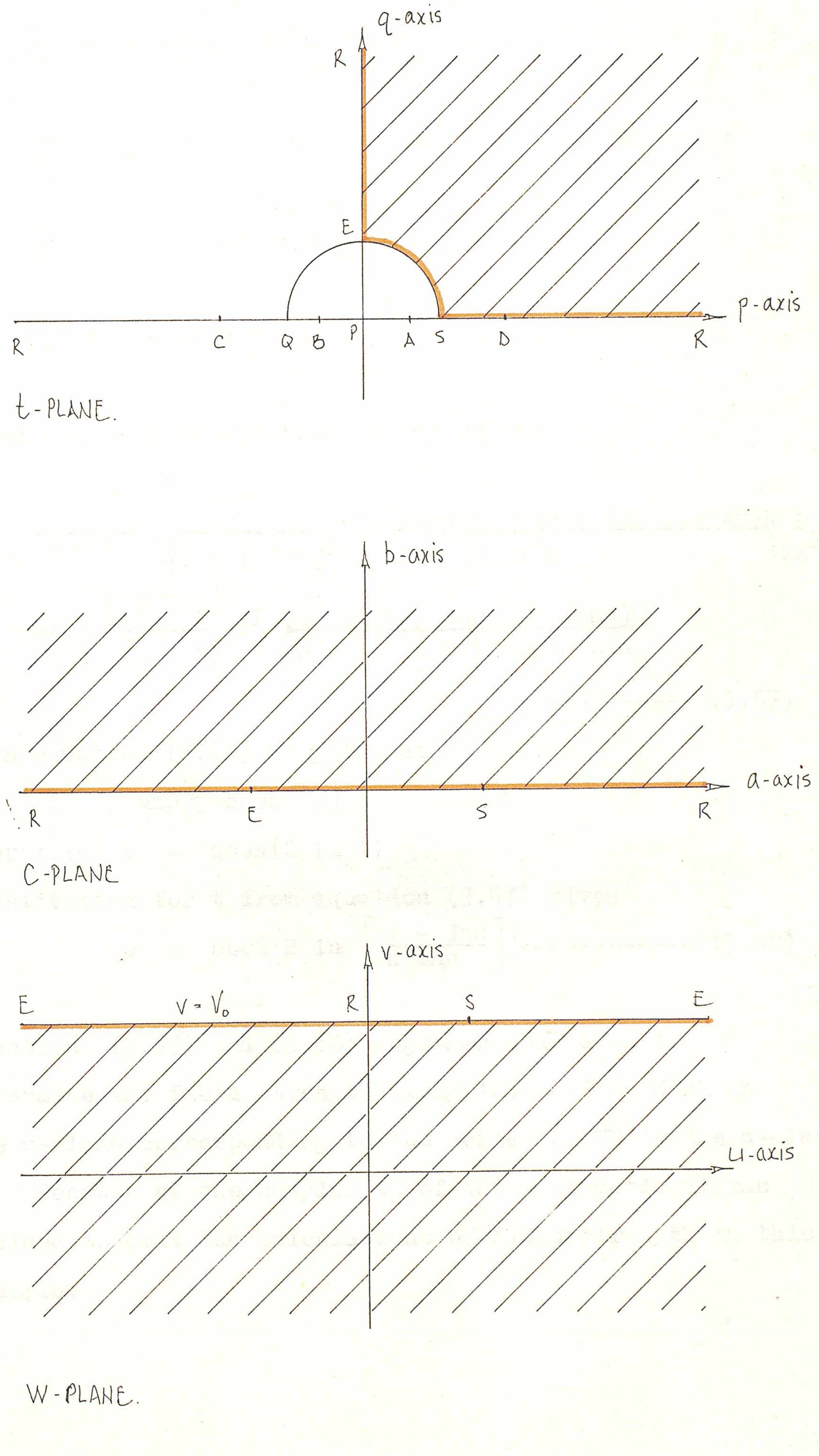


Figure 5.12 Electrical Transformation.

$$\frac{dz}{du} = -\frac{ibk^2}{L} \cdot cn^2u$$

$$\frac{dt}{du} = (k^2 \; cnu \; sn^2u - cnu \; dnu + cnu \; dn^2 \; u)/ksn^2u$$

$$\frac{dt}{dc} = \left[c + \sqrt{c^2 - 1}\right]^{\frac{1}{2}} / 2\sqrt{c^2 - 1}$$

$$\frac{dw}{dc} = -A/(c + 1)^2$$

Substituting into equation (5.66) we get

$$R = \left| \frac{A}{(c + 1)^2} \cdot \frac{2\sqrt{c^2 - 1}}{\left[c + \sqrt{c^2 - 1}\right]^{\frac{1}{2}}} \cdot \frac{(k^2 \; cnu \; sn^2u - cnu \; dnu \; (1 - dnu))}{k \; sn^2u} \cdot \frac{L}{ibk^2 cn^2u} \right|$$

$$= \frac{2A}{bk^3} \left| \frac{\sqrt{c^2 - 1}}{(c + 1)^2} \cdot \frac{L \; (k^2 \; sn^2u - dnu \; (1 - dnu))}{\left[c + \sqrt{c^2 - 1}\right]^{\frac{1}{2}} sn^2u \; cnu} \right|$$

------------ (5.67)

From equation (5.64) we get that

$$t = \exp\left[\tfrac{1}{2} \cosh^{-1} c\right]$$

Therefore $c = \cosh(2 \ln t)$

substituting for t from equation (5.63) gives

$$c = \cosh\left(2 \ln \left[\frac{1 - dnu}{k \; snu}\right]\right) \text{------------ (5.68)}$$

Equations (5.67) and (5.68) together enable us to determine the field strength along the surface SDRE in the u-plane corresponding to the corner ERDSE in the z-plane.

Because of the complexity of the arithmetic it was decided to omit the calculations of field strength in this example.

CHAPTER 6

ALTERNATIVE METHODS

It is an undeniable fact that most field problems that arise in practice cannot be solved by rigorous mathematics and some form of approximate method is required. Two such methods; the relaxation and experimental methods are commonly used in this situation.

6.1 Relaxation Method

This method which was originally invented by Southwell [20] in 1936 was at first used to determine stresses in a framework. The subsequent extension of the method to the solution of electric and heat field problems retains the original notation. For this reason the method will be introduced here using the mechanical analogy of a stressed framework.

Imagine a framework of rods held in position against a rigid background of constraints in the form of pegs at each junction. The framework is then loaded at various points, and will tend to deform under the action of these forces. Deformation however cannot take place because the pegs at the joints prevent this. Thus the loads are taken up by the pegs. Imagine now that the pegs are displaced in a convenient manner, determined by the conditions, so that the load is gradually transferred from the pegs to the framework. To do this, the peg which is taking the greatest share of the load is imagined to be moved so that the joint it is controlling is free to move so to relieve the strain. This transfers some of the load from the peg to the framework. The distribution of forces amongst the other constraining pegs will now be altered.

The joint which now, under the new force distribution, bears the greatest load is next displaced so that the load is lessened and some of it transferred to the framework from the peg. This process of imaginary movements of pegs is called the relaxation of the constraints. This second relaxation will redistribute the forces, and may even make the conditions at the first peg worse, but this does not matter as it is only temporary. The peg which now bears the greatest load is in turn displaced so that the constraint is relaxed at this point. This relaxation process is continued, going from peg to peg, always relaxing that joint which at any moment bears the greatest load.

This is called the 'systematic relaxation of the constraints' and is carried out until the load is completely transferred from the pegs to the framework. At the end of this relaxation process, the framework, correctly deformed, will be carrying the whole load, so that the pegs at the joints can be removed. At any intermediate stage the pegs will be bearing a part of the load. This value is called the residual at that point. When all the residuals are reduced to zero, all the forces will have been removed from the pegs. The object of the systematic relaxation of the constraints is therefore to reduce the residuals at all points in the system to zero, or as near to zero as required. The residuals can be made as small as we like by carrying out the relaxations a sufficient number of times, and thus any degree of accuracy can be attained.

In the case of electric or heat fields we require a solution of Laplace's equation in two dimensions which is

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{------------ (6.1)}$$

where w would be electric potential in the case of electric fields or temperature in the case of heat fields. Instead of carrying out a physical relaxation in the manner described above, the same procedure is carried out mathematically. At each point in the imaginary framework we have an equation connecting w with the position at that point. Thus there is an equation for each point in the meshwork. All these equations must obey the governing equation of the system, namely Laplace's equation. With the field determined at each point, the points of equal magnitude can be joined by curves representing isothermals, equipotentials or whatever is appropriate to the system.

It must be pointed out that the system of equations we have been referring to must all be solutions of Laplace's equation. Since the relaxation method refers only to linear algebraic equations, an essential part of the problem is the conversion of Laplace's equation into an equivalent linear equation of the right type. One way of doing this is to use Taylor's Theorem as described by Allen [21].

Representation of Laplace's Equation on a square Lattice.

Figure 6.1 represents a square lattice of side h in a two-dimensional electric field. At any corner point of

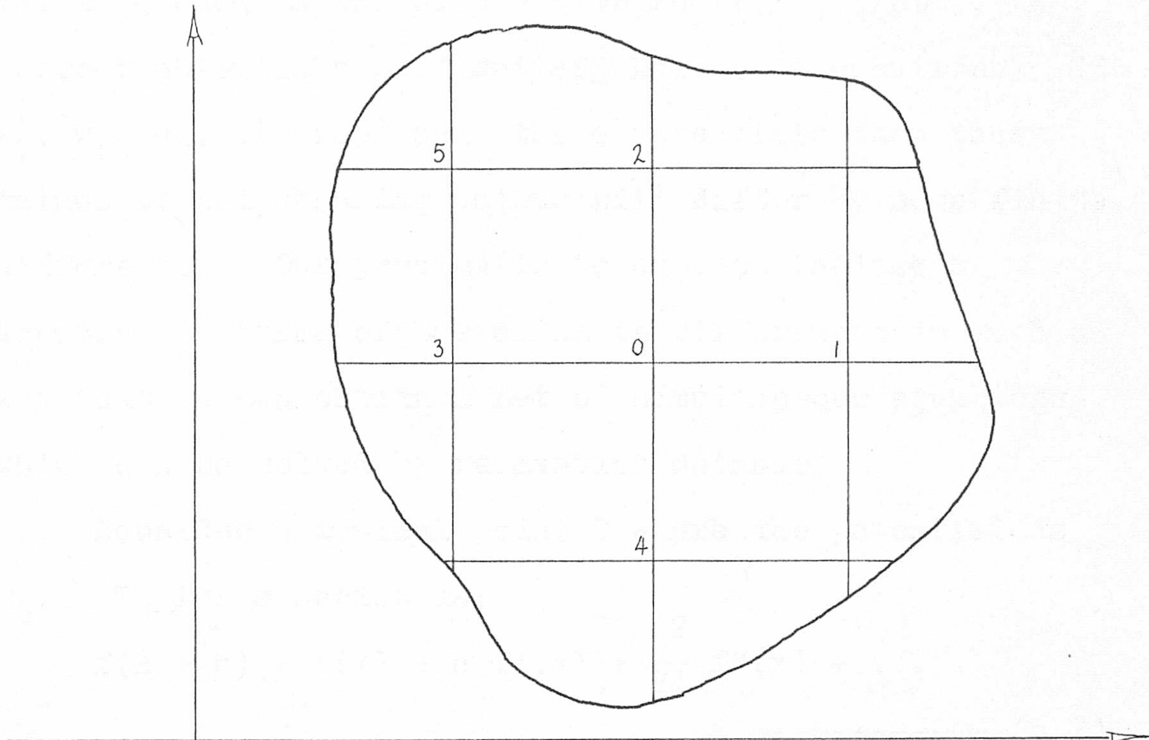


Figure 6.1 Field mesh of side h.

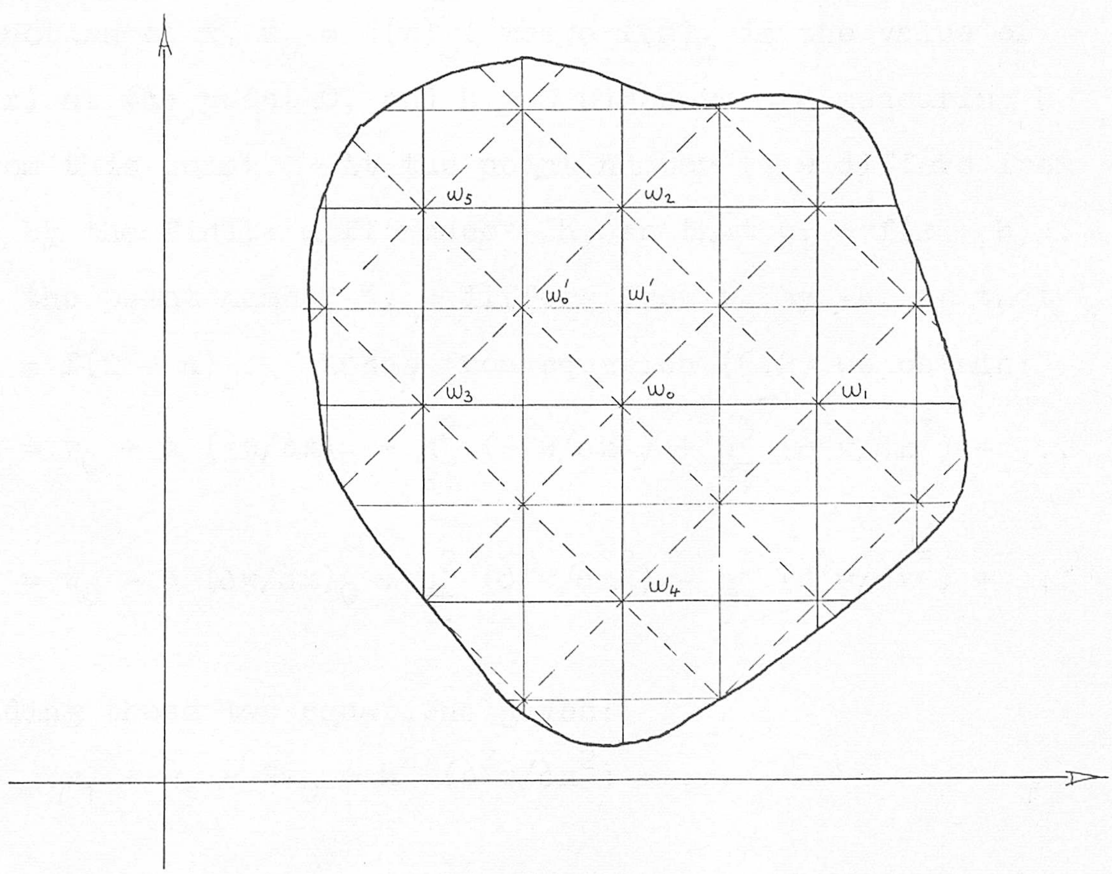


Figure 6.2 Field mesh of side h/2.

the mesh such as the points numbered 1, 2, 3 etc., the correct potentials will satisfy Laplace's equation. If w_0, w_1, w_2, etc represent these potentials then these values at neighbouring points will differ by some finite difference. Our problem is to express Laplace's equation in terms of these finite differences in such a way that we can obtain a set of simultaneous equations which can be solved by relaxation methods.

Consider a typical point 0 where the potential is w_0. Taylor's series is:

$$f(x + h) = f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \ldots \qquad \text{(6.2)}$$

Consider now the points 0, 1 and 3, that is the potential variations along the direction of the x-axis only. If we take the point 0 as the origin, then, since w is a function of x, $w_0 = f(x)_0$, where $f(x)_0$ is the value of f(x) at the point 0, and h = 0 since we are measuring h from this point. At the point number 1, w differs from w_0 by the finite difference + h, so that $w_1 = f(x + h)_0$. At the point number 3, w differs from w_0 by -h, so that $w_3 = f(x - h)_0$. Hence from equation (6.2) we obtain:

$$w_1 = w_0 + h\,(\partial w/\partial x)_0 + \frac{h^2}{2!}(\partial^2 w/\partial x^2) + \frac{h^3}{3!}(\partial^3 w/\partial x^3) + \ldots$$

$$w_3 = w_0 - h\,(\partial w/\partial x)_0 + \frac{h^2}{2!}(\partial^2 w/\partial x^2) - \frac{h^3}{3!}(\partial^3 w/\partial x^3) + \ldots$$

Adding these two equations gives:

$$w_1 + w_3 = 2w_0 + h^2\,(\partial^2 w/\partial x^2) + \ldots$$

If we denote the terms greater than h^2 by $F(h^4)$ then:

$$h^2\ (\partial^2 w/\partial x^2) = w_1 + w_3 - 2w_0 - F(h^4)$$

If h, which is the mesh side length, is small, then as an approximation we can write:

$$h^2\ (\partial^2 w/\partial x^2) = w_1 + w_3 - 2w_0 \quad \text{------------} \quad (6.3)$$

If the same reasoning is repeated when the variation of w is taken along the y-axis, so that we consider the points numbered 0, 2 and 4, we can obtain:

$$h^2\ (\partial^2 w/\partial y^2) = w_2 + w_4 - 2w_0 \quad \text{------------} \quad (6.4)$$

again with an error of $F(h^4)$. Adding equations (6.3) and (6.4) gives:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{h}\left[w_1 + w_2 + w_3 + w_4 - 4w_0\right] \quad \text{----} \quad (6.5)$$

Now if Laplace's equation is satisfied by the value w_0 at the point 0, the left hand side of equation (6.5) becomes zero. Hence if

$$w_1 + w_2 + w_3 + w_4 - 4w_0 = 0$$

the w-values at the five points considered must be correct, ignoring the error term $F(h^4)$. If one or more of the potentials w are incorrect then:

$$w_1 + w_2 + w_3 + w_4 - 4w_0 = R_0 \quad \text{------------} \quad (6.6)$$

where R_0 is not zero. Thus R_0 is the residual at the point 0, and the equation we have obtained is the algebraic equation connecting the w-values for the group of five points numbered 1, 2, 3, 4 and 0. There is a similar equation for each point in the field, and the residuals R_0, R_1, R_2 etc for each of these points must be reduced to zero to obtain the correct potential distribution.

It should be noted that if we find the correct values of w_0, w_1, w_2, w_3 and w_4 so that R_0 is zero, this is only a temporary solution as these values, while being correct for the point 0, will not necessarily be correct for the field as a whole. The residual with the point w_1 as the central point of a new group five points must then be reduced to zero and this will affect its neighbouring points, and so on.

In figure 6.1 we see part of the field of a conductor system. The conductor surfaces provide the boundary values of the system which must be marked on the diagram. Values of potential at the mesh points are then guessed and written at each point so that an approximate field is established. Taking each point in turn the residuals are calculated using equation (6.6) and written beside the point. The process of relaxation can then begin.

When the residuals are all zero, or as close as required, the mesh can be reduced by drawing diagonals through the corner points, seen in figure 2 as dotted lines. The new mesh points are formed at the intersection of these diagonals. By drawing vertical and horizontal lines through these points the mesh is reduced to side h/2. Figure 6.2 shows the new mesh point w_0' formed in this way. The function value at this point is equal to the arithmetic mean of the four surrounding mesh points, in this case w_3, w_0, w_2 and w_5. Any remainder in the calculation becomes the residual. Corner point w_1' is equal to the mean of w_0 and w_2, and so on.

The process of relaxation can now begin again giving a more accurate picture of the field. The degree of accuracy required will determine the number of reductions and when that has been reached the lines of equipotential can be drawn in.

6.2 Experimental Method

By far the most useful experimental method employs electric conduction. For a plane distribution we may measure the potential in a sheet of tinfoil or specially coated conducting paper. The electrodes are simulated by heavy copper plates made to scale. If the system consists of two or more plates at different potentials then these plates are simply soldered on the tinfoil. If only one plate exists then another plate must be introduced at some considerable distance from the first to act as a terminus for the lines of force. Since the lines of force are supposed to terminate at infinity, it is desirable that the second plate lies along a known equipotential.

The general arrangement of a two conductor system is shown in figure 6.3. The electrodes are connected to a battery and then to a pair of resistance boxes which act

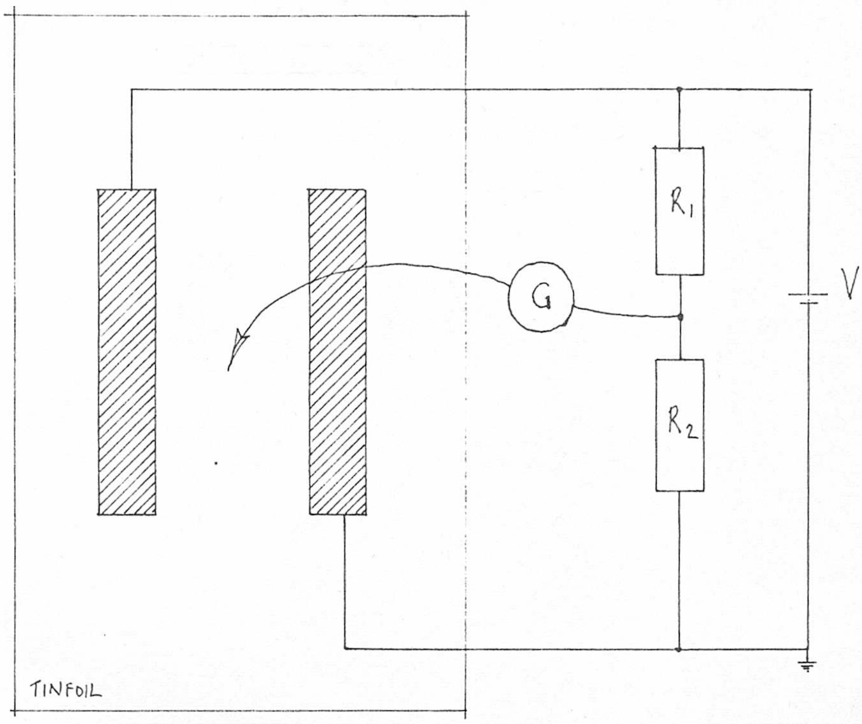


Figure 6.3 Experimental set up for two conductor system.

as a potential divider. If $R_1 = R_2$ then the potential of the probe when the latter is not in contact with the tinfoil will be midway between the two electrodes. The probe is now touched to the sheet and moved about until the galvanometer G_1 reads zero. This point will be on the equipotential line V/2. Other points on this line can be found similarly. The points can be marked directly on the tinfoil or the probe can be attached to a pantograph so that the equipotentials can be transferred to a drawing. The ratio R_1/R_2 is then changed allowing other equipotentials to be mapped.

The accuracy of the method is dependent among other things on the uniform thickness of the tinfoil and the quality of the connections between the plates and the tinfoil

CHAPTER 7

CONCLUSIONS

We have used the method of Conformal Transformations to investigate successfully several corner configurations and illustrated the various methods of solution most commonly employed. In conclusion it is necessary to say something about the limitations of the method and particularly the difficulties encountered in some of the problems.

In the attempted analysis of the corner in figure 7.1, it was initially decided to use the Richmond Method but a discontinuity at the point D made this unsuitable. When the transformation is made from the z to the t-planes the

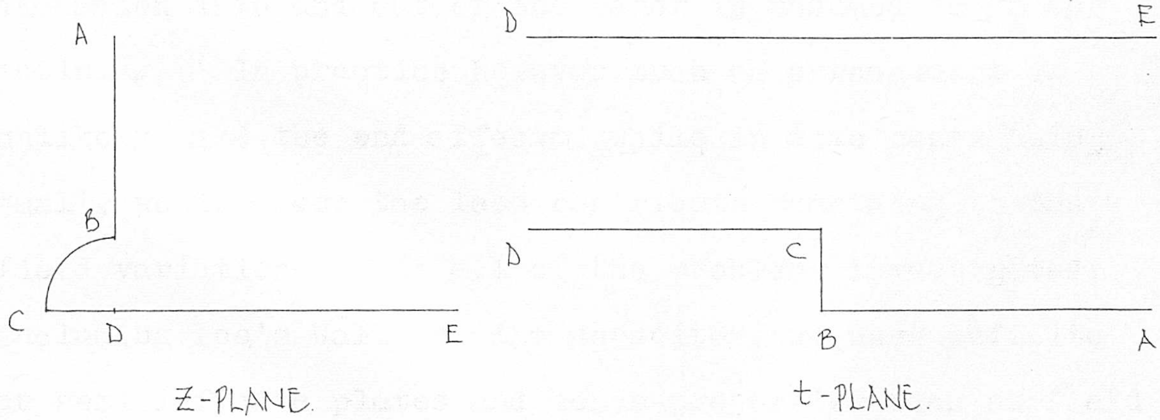


Figure 7.1

points A and D in the z-plane both transform to infinity in the t-plane. Consequently when using the Richmond Method it is necessary to avoid impinging the unit circle. Section 5.2 shows the two corner configurations that can be solved successfully with this method.

The use of the integral equations of the hypergeometric series is limited to our ability to integrate them. When the substitution of elliptic functions is made it is found that when the powers to which the functions are raised are integers, then the integral can usually be evaluated, although integers greater than three are excessively complicated. In most cases fractional powers are not solvable. This means that there must be a limited number of problems that can be evaluated by this method; a point more fully examined by Langton [22].

The Schwarz-Christoffel transformation is again dependent on our ability to evaluate the integral but is probably the most useful method.

As was stated in Chapter 1, conformal transformations are suitable for two-dimensional fields where the third dimension into and out of the paper is assumed to go to infinity. In practice however such an arrangement is unlikely, and the end effects, while in some cases being small, would never the less contribute something to the field variations. In all of the problems investigated, including Lee's Wall and the capacitor, we used infinite or semi-infinite plates and hence properties such as field strength only reach their uniform ideal values at infinity. Again in practice uniformity would be reached after a finite distance and for all practical purposes the effects of a variation in the configuration of a corner would become negligible at a distance less than that indicated in theory.

We are faced here with the inevitable consequences of sacrificing physical reality for mathematical simplicity. In our attempts to obtain an exact mathematical solution we have introduced such dubious concepts as infinite plates and perfect corners. Having said this we must assume that this investigation of the properties of various two-dimensional fields will yield results that are mathematically accurate but only approximate in practical cases, because the end effects prevent actual fields from being strictly two-dimensional.

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